NeoFregeanism and Quantifier Variance*

Theodore Sider *Aristotelian Society, Supplementary Volume* 81 (2007): 201–32

NeoFregeanism is an intriguing but elusive philosophy of mathematical existence. At crucial points, it goes cryptic and metaphorical. I want to put forward an interpretation of neoFregeanism—perhaps not one that actual neoFregeans will embrace—that makes sense of much of what they say. NeoFregeans should embrace *quantifier variance*.[1]

1. NeoFregeanism

The neoFregeanism of Bob Hale and Crispin Wright is an attempt to resuscitate Frege's logicism about arithmetic. Its goal is to combine two ideas. First: platonism about arithmetic. There really do exist numbers; numbers are mind-independent. Second: logicism. Arithmetic derives from logic plus definitions. Thus, arithmetic knowledge rests on logical knowledge, even though its object is a realm of mind-independent abstract entities.

1.1 Frege on arithmetic

Let us review Frege's attempt to derive arithmetic from logic plus definitions. "Arithmetic" here means second-order Peano arithmetic. "Logic" means (impredicative) second-order logic.[2] The "definition" is what is now known as *Hume's Principle*:

Hume's Principle $\forall F \forall G(\#x{:}Fx=\#x{:}Gx \longleftrightarrow \mathrm{Eq}(F,G))$

*Matti Eklund's work connecting neoFregeanism to questions about the ontology of material objects (2006*b*, 2006*a*) sparked my interest in these topics. Thanks to Matti for helpful comments, and to Frank Arntzenius, Deniz Dagci, Kit Fine, John Hawthorne, Eli Hirsch, Anders Strand, Jason Turner, Dean Zimmerman, attendees of the 2006 BSPC conference (especially Joshua Brown, my commentator), and participants in my Spring 2006 seminar on metaontology. This version corrects a few errors in the published version; thanks to Jared Warren and Tim Williamson.

[1]I recommend quantifier variance to neoFregeans about nonmathematical ontology as well (e.g., Schiffer (2003)). The term 'quantifier variance' is from Hirsch (2002*b*).

[2]See Boolos (1985) on the relationship between Frege's original logical system and contemporary systems.

"The number of Fs = the number of Gs iff F and G are equinumerous"

'$\#x{:}\phi$' is to be read as "the number of ϕs". (Grammatically, '#' is a monadic predicate functor; it combines with a variable and a single open sentence to yield a term.) '$\mathrm{Eq}(F,G)$' (read "F and G are equinumerous") abbreviates the following formula:

$$\exists R[\forall x \forall y \forall z \forall w([Rxy \wedge Rzw] \rightarrow [x{=}z \longleftrightarrow y{=}w]) \wedge \forall x(Fx \rightarrow \exists y[Gy \wedge Rxy]) \wedge \forall y(Gy \rightarrow \exists x[Fx \wedge Rxy])]$$

"There exists a one-to-one correspondence between the Fs and the Gs"

Now, Frege himself did not regard Hume's Principle as a *definition* of the expression 'the number of'. Taken as a definition of the left-hand side of its biconditional, Hume's Principle would define only sentences of the form 'the number of ϕs = the number of ψs'. It would be inapplicable to sentences of other forms, such as 'The number of ϕs = Julius Caesar'. Any "definition" of number that doesn't settle whether Julius Caesar is a number is no definition at all. Instead, Frege defined numbers as the extensions of certain concepts (these extensions were in essence sets). Given his background theory of extensions, Frege was able to derive Hume's Principle as a theorem, and went on to derive the axioms of second-order Peano Arithmetic.

1.2 From Fregeanism to neoFregeanism

Infamously, Frege's underlying theory of extensions was subject to Russell's paradox. Unable to repair the inconsistency in his system that Russell pointed out to him, Frege eventually came to regard his logicism as a failure. It was only noticed much later that Frege's derivation of arithmetic relied on the inconsistent theory of extensions at only one place: in the proof of Hume's Principle.[3] After that point, the extensions were no longer needed; the derivation of arithmetic subsequently relied only on Hume's Principle. That raised the possibility that Hume's Principle itself, unlike Frege's theory of extensions, is *not* inconsistent. This was indeed shown to be the case.[4]

[3] The point was first made in passing by Charles Parsons (1965), and was later independently made by Crispin Wright (1983), who emphasized its philosophical significance.

[4] Hume's principle has been proved to be consistent with second-order logic, relative to the consistency of systems that everyone believes to be consistent (Hazen, 1985; Burgess, 1984).

This mathematical result—that Peano arithmetic (under appropriate definitions) follows in second order logic from Hume's Principle, which is consistent—is remarkable indeed, but on its face is no vindication of logicism. Since Hume's Principle implies the truth of arithmetic, it implies the existence of infinitely many things. How can something that implies the existence of even one thing, let alone infinitely many, be a definition?

What the neoFregeans claim is that, despite its existential implications, Hume's Principle is nothing more than a definition of number, and therefore can be known to be true a priori.[5] Thus, in a sense, objects may be introduced by definition.[6]

Hume's Principle isn't an *explicit* definition of number, since as noted above, it doesn't apply to all linguistic contexts containing 'the number of'. Nevertheless, neoFregeans say, it is an *implicit* definition: it defines the expression 'the number of' by stipulating how that expression is to perform in *some* linguistic contexts. Think of the act of laying down the definition as the delivery of instructions to the semantic gods: "let my expression 'the number of' be so understood as to obey Hume's principle".

But of course, this just invites the question of whether there *is* any way to understand 'the number of' so that Hume's Principle comes out true.[7] If there do exist infinitely many objects, then perhaps there is a way[8], but if there are not, one wants to say, there may simply be no way of interpreting 'the number of' so that Hume's Principle comes out true. In that case, the semantic gods will respond to our instructions with a blank look, as they would (assuming atheism) if we stipulated that 'God' is to denote the omnipotent being who created the world (Field, 1984).

It is in response to this worry that neoFregeanism becomes fascinatingly and maddeningly obscure. I will discuss two lines of thought: "the priority of syntax" and "reconceptualization".

[5] The leading defenders are Crispin Wright and Bob Hale. See Wright (1983); Hale (1987); Hale and Wright (2001).

[6] It is this aspect of neoFregeanism that has most perplexed commentators (see, for instance, Field (1984)), and it is this aspect on which I will focus. I will set aside the "Julius Caesar problem", but see Rosen (1993) for some promising ideas.

[7] It also invites the question of whether there is more than one way; this, in essence, is the Caesar problem.

[8] Though even here the way may not be straightforward. Hume's Principle implies the existence of an entity that is the number of absolutely all objects; the numbers of which it speaks cannot, therefore, be the cardinals of standard ZF set theory. See Boolos (1997).

1.3 The priority of syntax

NeoFregeans want to reassure us that there is indeed a way of taking 'the number of' so that Hume's Principle comes out true. What if they just stipulated that '$\#x{:}Fx = \#x{:}Gx$' is to *abbreviate* '$\mathrm{Eq}(F,G)$'? Hume's Principle would then certainly come out true; it would abbreviate the logical truth '$\forall F \forall G(\mathrm{Eq}(F,G) \longleftrightarrow \mathrm{Eq}(F,G))$'.

But you can't both stipulate that a complex string of symbols is to abbreviate something, and also treat the string's parts as semantically significant constituents. (If you coin a new name for me, 'AsonofaBush', you can't infer from the resulting truth of 'Ted is AsonofaBush' that I am related by birth to the esteemed president of the United States.) NeoFregeans certainly do assume that 'x', '$=$', 'F', and 'G' are semantically significant constituents of '$\#x{:}Fx = \#x{:}Gx$'; the derivation of Peano arithmetic from Hume's Principle depends on it. So they can't just say that '$\#x{:}Fx = \#x{:}Gx$' abbreviates '$\mathrm{Eq}(F,G)$'.

But real live neoFregeans say something that is almost this. Their position depends on the legitimacy of making *both* of the following stipulations:

(1) '#' is to be understood so that it obeys Hume's Principle

(2) $\ulcorner \#x\phi \urcorner$ thus understood is to have the logical form it appears to have (e.g., existential generalization on the entire expression $\ulcorner \#x\phi \urcorner$ is valid; e.g., the constituent expressions of ϕ are semantically significant, so that, for instance, variables in ϕ may be bound to external quantifiers)

They defend the propriety of jointly making these stipulations by saying that there is nothing more to having a certain logical form beyond having an appropriate syntactic distribution throughout the language. If an expression can occur grammatically in the sorts of places that a singular term can occur, then it *is* a singular term. Wright calls this view the "priority of syntactic over ontological categories"; here is a representative quotation (1983, 51–52):

> According to [the thesis of the priority of syntactic over ontological categories], the question whether a particular expression is a candidate to refer to an object is entirely a matter of the sort of syntactic role which it plays in whole sentences. If it plays that sort of role, then the truth of appropriate sentences in which it so features will be sufficient to confer on it an objectual reference; and questions concerning the character of its reference should then be addressed by philosophical reflection on

> the truth-conditions of sentences of the appropriate kind. If, therefore, certain expressions in a branch of our language function syntactically as singular terms, and descriptive and identity contexts containing them are true by ordinary criteria, there is no room for any ulterior failure of 'fit' between those contexts and the structure of the states of affairs which make them true. So there can be no philosophical science of ontology, no well-founded attempt to see past our categories of expression and glimpse the way in which the world is truly furnished.

Thus, we can stipulate that the truth condition of '$\#x{:}Fx = \#x{:}Gx$' is simply $\mathrm{Eq}(F, G)$ without forfeiting the status of '$\#x{:}Fx$' as a genuine singular term.[9] It is a genuine singular term because it occurs grammatically in the places where genuine singular terms can occur.

There is a complex literature on what a purely syntactic criterion for being a "genuine singular term" might be. But the notion of a singular term is relevant to the current debate only because the neoFregean wants to employ the usual quantificational laws to sentences containing $\ulcorner \#x{:}\phi \urcorner$. So instead of fighting over what it means to be a genuine singular term, I suggest simply *stipulating* that '#' is to create terms with semantically significant constituents to which the usual quantificational laws apply. The debate can then focus on the real issue: whether the joint stipulation of (1) and (2) is coherent. This stipulation costs the neoFregean nothing and exposes the dispute over syntactic priority as distracting noise.

1.4 Reconceptualization

What guarantee is there that (1) and (2) may be jointly stipulated? Here we reach the crux of the issue. Wright says, in effect, that the state of affairs of F's being equinumerous to G can be "reconceptualized" as a state of affairs involving the existence of numbers. Or better: the totality of states of affairs about equinumerosity can be reconceptualized as states of affairs involving the existence of a domain of numbers. Wright makes the point with a different example, that of reconceptualizing states of affairs about parallel lines as states of affairs involving the existence of directions:[10]

> Consider again the abstraction for directions:

[9] Focus on the category of singular terms is a bit misplaced, for in addition to '$\#x{:}Fx$' being a singular term, its parts must also be semantically significant.

[10] Wright (1997), pp. 277–278 (pagination from Hale and Wright (2001).)

$Da = Db$ if and only if $a//b$.

The dilemma was that we either regard the left hand side simply as a definitional transcription of the right, and thereby forfeit the possibility of taking its syntax at face value, of treating it as a genuine identity statement linking genuine singular terms in existentially generalizable position; or we take the principle as a substantial claim, to the effect that certain abstract objects—directions—are associated with lines in the way it describes, in which case we have no right simply to lay the principle down as a definition. But the key to Frege's view is that the dilemma is a false one—it is the thought, roughly, that we have the option, by laying down the Direction abstraction, of *reconceptualizing*, as it were, the type of state of affairs which is described on the right. That type of state of affairs is initially given to us as the obtaining of a certain equivalence relation—parallelism—among lines; but we have the option, by stipulating that the abstraction is to hold, of so reconceiving such states of affairs that they come to constitute the identity of a new kind of thing, directions, of which, by this very stipulation, we introduce the concept. The concept of direction is thus so introduced that that two lines are parallel *constitutes* the identity of their direction. It is in no sense a further substantial claim that their directions exist and are identical under the described circumstances. But nor is it the case that, by stipulating that the principle is to hold, we thereby forfeit the right to a face-value construal of its left-hand side, and thereby to the type of existential generalization which a face-value construal would license. When the abstraction principle is read in the way which Frege proposes, its effect is so to fix the concept of direction that there is absolutely no gap between the existence of directions and the instantiation of properties and relations among lines.

It is important to be clear that it would be a misrepresentation of this idea to view it as involving the notion that abstract objects are creations of the human mind, *brought into being* by a kind of stipulation. What is formed—created—by such an abstraction is rather a *concept*: the effect is merely to fix the truth-conditions of identity statements concerning a new kind of thing, and it is quite another question whether those truth-conditions are ever realized. If we accept the concept-formation involved in the Fregean abstraction of Direction, the effect is not to define directions into existence but to coordinate the question of the existence of directions with that of the existence of lines; and the latter can remain, for all that is implicit in an acceptance of the abstraction, as objective and mind-independent a matter as you want.

If this idea of reconceptualization makes sense, then the question of whether 'the number of' forms "genuine singular terms" is irrelevant, for the neoFregean can say that states of affairs concerning equinumerosity may be reconceptualized so as to admit of characterization using "singular terms" in a sense stipulated to obey the usual quantificational laws. But this notion of reconceptualization is notoriously obscure. What exactly does it amount to?

My proposal: quantifier variance. There are many equally good things one can mean by the quantifiers. If on one 'there are numbers' comes out false, there is another on which 'there are numbers' comes out true. The facts are the same either way; it's just that the facts have no unique description using quantificational language. Compare the sense in which facts about measurable quantities have no unique description, given the arbitrariness of the choice of a unit of measure. We could use the expression 'one meter' so that 'this bar is one meter long' comes out true, or we could use 'one meter' so that it comes out false. Neither linguistic choice is better than the other. Similarly for quantificational language. "Reconceptualization" means selecting a meaning for the quantifiers on which Hume's Principle comes out true.

2. Quantifier variance

2.1 NeoCarnapianism

Quantifier variance is the position of Rudolf Carnap's (1950) contemporary soulmates, who want to deflate philosophical debates over ontology.[11]

Consider, for example, the debate over the ontology of composite material objects. When are given material objects part of some further composite object? Some say: *always*. There exist scattered objects. Some say: *never*. No composite material objects exist. Some say *sometimes*. If objects are appropriately glued together (or whatever) then there exists a further material object that they compose, otherwise not.[12]

The neoCarnapians recoil from all this in horror. Their guiding thought is that nothing is really at issue in this so-called debate, beyond how to talk. Now, the folks I have in mind reject Carnap's positivism (hence the 'neo'). In place of Carnap's linguistic frameworks, they have different languages. The languages correspond to different decisions about what quantificational expressions are

[11] I have in mind primarily Eli Hirsch (2002*a*,*b*, 2005, 2007); see also Hilary Putnam (1987*a*,*b*).

[12] The issue was thus framed by van Inwagen (1987, 1990).

to mean. In the different languages, quantificational sentences like ‘there exist tables’ express different propositions. In one language, ‘there are tables’ is true; in another, ‘there are no tables’ is true. Which proposition is expressed by a given sentence is of course a matter of convention; but the truth of the proposition itself can be as mind-independent and evidence-transcendent a matter as you like.[13]

The view is intended to be ontologically deflationary because:

i) For each competing theory about the ontology of composite material objects, quantificational expressions can be interpreted so that the theory comes out true

ii) None of these interpretations is any “better” than the others

Now, as I see it, in order to secure ii), the quantifier variantist must hold that none of the interpretations is a more *natural* interpretation than the others; none “carves logical reality at its joints” better than the others; no one is most “basic” or “fundamental”.[14] For if one distinguished interpretation were more natural than the others, then the ontological debate could continue undeflated: as a debate about what exists in the distinguished sense. So as a first pass, I formulate quantifier variance—as applied to the debate over composite material objects—as follows:[15]

NeoCarnapian quantifier variance: There is a class, C, containing *equally* and *maximally* natural candidate meanings for quantifier expressions, in that: i) no member of C is more natural than any other member of C, and ii) no candidate meaning for quantifier expressions that is not in C is as natural as any

[13] NeoCarnapianism had better not collapse into the banal claim that since all language is conventional, any sentence, construed as a bare string of symbols, can be interpreted truly. The neoCarnapian’s languages are not supposed to be utterly semantically *alien* (compare the discussion of conventionality in Sider (MS)). In each case, ‘there exists’ is to count as “a kind of quantifier”, one might say. But it can’t be “a kind of quantifier” in the most straightforward sense; see section 2.2. The various interpretations of ‘there exists’ must count as being similar to one another, but in what way? Hirsch’s (2002*b*, p. 53) suggestion is that they must share an appropriate inferential role.

[14] The notion of naturalness I have in mind is a generalization of Lewis’s (1983; 1984; 1986, pp. 59–69); see Sider (2009) and my forthcoming book for an extended discussion of its application to metaontology and other questions of metametaphysics.

[15] I would cash out ‘candidate meaning’ in terms of inferential role; see note 13.

member of C. Each position in the debate over the ontology of composite material objects comes out true under some member of C.

2.2 How not to refine quantifier variance

What *are* these candidate quantifier meanings? The most straightforward characterization does not work. The most straightforward characterization is that the candidates result from choosing different domains for the quantifiers. To say this, we ourselves would need to quantify over all the objects in all the domains—we would be saying that *there is* a domain containing *all* of the objects over which the quantifiers of L range, for various languages L. But the language we're speaking might be one of the languages in question, and not the "biggest" one.

Might the quantifier variantist stick to the straightforward characterization by i) saying that the quantifiers in each of the "smaller" languages are restrictions on the quantifiers in a single, biggest language; and ii) admitting that the doctrine of quantifier variance can only be stated in this biggest language?[16] No: this would undermine the quantifier variantist's egalitarianism. When one restricts a quantifier, one simply *ignores* some of the things to which one is committed. When pressed on whether the ignored things exist, one *ought* to undo the restriction and admit that the things exist after all. In a conversation with a biologist who is pointing out the existence of microbes, air, and the like, it would be *wrong*—conversationally and epistemically—to dig in one's heels and insist that there is absolutely nothing in the refrigerator. Indeed, there is something epistemically *superior* about the context in which one agrees that there are some things in the refrigerator. But the quantifier variantist does not want to say that, if one is speaking a language that eschews scattered objects, one ought to admit under pressure that the scattered objects exist after all, or that the epistemic position of the user of the more inclusive quantifier is superior. That would be taking sides on the first-order debate.

Further, recall that the quantifier variantist thinks that the meanings corresponding to the various positions on the ontology of composite objects are all equally and maximally natural. But surely if Q^- results from restricting a maximally natural quantifier meaning, Q, then Q^- is *less* natural than Q.

So: the quantifier variantist cannot characterize the various quantifier mean-

[16] One objection I will not press is that there may be no biggest language.

ings as corresponding to different domains. How, then, can those meanings be characterized?

2.3 Meanings and contexts[17]

On behalf of quantifier variantists, I will take an "algebraic" approach to quantifier meanings. Rather than trying to specify what these meanings *are* intrinsically, I will specify only what they are supposed to *do*. I will introduce a space of quantifier meanings endowed with enough structure to do the work that quantifier variantists want done.

Quantifier variantists might fill in this structure in different ways. Some might, for instance, construe a quantifier meaning as a way of translating quantified sentences into some chosen language. Others might construe quantifier meanings as possible-worlds truth conditions of quantified sentences. Others might take quantifier meanings to be sui generis entities. Still others might be fictionalist about talk of meanings.[18] The algebraic approach gives merely the minimal structural commitments of quantifier variantism.

What are the quantifier meanings supposed to do? First, the notion of naturalness (carving at the joints) must apply to them. Second, they are to (help) determine truth values for quantified sentences. Third, it must make sense to speak of more or less "expansive" quantifier meanings, where this is not merely a matter of varying domain restrictions.

So, let us speak of entities called *meanings*. Think of a "meaning" as being a meaning for a whole language, though our primary focus is the quantifiers. In addition, to account for contextual variation of quantifier domains, let us speak of further entities called *contexts*. To the meanings and contexts let us apply the following undefined predicates:

meaning m is *at least as natural as* meaning m'

context c *belongs to* meaning m

model M *depicts* meaning-context pair $\langle m, c\rangle$

The first predicate is used to measure how well meanings carve nature at the joints. The second predicate is needed to attach the contexts to the meanings ("contexts" are supposed to be contexts of utterance for quantified

[17]Those impatient with the following details may skip ahead to section 2.7 with little loss.

[18]See Turner (2008) for an interesting further approach.

sentences given a certain meaning for the quantifiers.) Call a meaning-context pair $\langle m, c\rangle$, where c belongs to m, a *quantifier*; the idea of the third predicate is that the "world according to a quantifier"—what the domain of existing objects "looks like" from the perspective of that quantifier—can be depicted by a model. Given section 2.2, we cannot think of these models as *intended* models—models whose domains are the intended domains of quantification, under the quantifier meanings. That would turn each of the candidate quantifier meanings into restrictions of the quantifier used to formulate quantifier variantism. "Models" here are *not* intended models; they're just models in the sense of model theory, in which the domains are allowed to contain any old objects.[19]

It is natural to make the following assumptions about the meanings, contexts, and primitive predicates we have applied to them:

- Each context belongs to exactly one meaning
- No model depicts anything other than a quantifier
- Each quantifier is depicted by some model
- The same sentences are true in any two models that depict the same quantifier

Further, we can define a notion of truth for a sentence relative to a given meaning (in a given context):

Definition of truth Sentence ϕ is $true^c_m$ iff ϕ is true in some model that depicts $\langle m, c\rangle$

We have seen how to use our meanings, contexts, and primitive predicates to do two of the three things that quantifier meanings are supposed to do: speak of naturalness of meanings, and of sentences being true relative to meanings. I show how to do the third thing—speak of more or less expansive quantifier meanings—in the next section.

The algebraic approach quantifies over meanings, sentences, and sets. But quantifier variantism might apply even to quantification over these entities. Does this threaten the approach?

[19]When M depicts q, which language M interprets depends on which sentences q interprets. I will assume that all the sentences to be considered have the usual syntactic categories: quantifiers, variables, predicates, etc. When we get to arithmetic (below), some of the sentences will be second-order.

I don't think so. Consider a few characters. An *opponent of quantifier variantism who accepts abstracta* can clearly adopt the approach. A *proponent of quantifier variantism* can also adopt the approach while speaking one of her languages that allows quantification over abstracta. Admitting that a "large" language must be used to state quantifier variance is not an embarrassment, for it does not imply that the quantifier of the large language carves nature at its joints any better than do the quantifiers of smaller languages.[20] A *nominalist opponent of quantifier variance* is the character who is most likely to encounter trouble. But surely *some* way (fictionalist or otherwise) can be found to make talk of abstract entities nominalistically acceptable—and if not, the inability to formulate quantifier variance would be the least of the nominalist's worries.

2.4 Expansion and restriction

Everyone agrees that one can "shrink" the domain of quantification: by quantifier *restriction*.[21] The distinctive claim of quantifier variantism, on the other hand, is that quantifiers can be in some sense *expanded*; and as we saw in section 2.2, this expansion is not the mere removal of restrictions. We must characterize this distinctive claim within the algebraic approach. The rough idea is: mere restriction changes the context but retains the same meaning, whereas the distinctive kind of expansion changes the meaning as well as the context.

More precisely, consider the following definitions:[22]

> **Def of supermodel** Model M ($=\langle D, I\rangle$) is a *supermodel* of model M$'$ ($=\langle D', I'\rangle$) iff i) $D' \subseteq D$, ii) $I(\alpha) = I'(\alpha)$ whenever $I'(\alpha)$ is defined, for each name α, iii) $I(\pi) \cap D'^n = I'(\pi)$ whenever $I'(\pi)$ is defined, for each n-place predicate π, and iv) $I(\nu)(U) = I'(\nu)(U)$ for all $U \subseteq D'^n$ whenever $I'(\nu)$ is defined, for each n-place predicate functor ν. M is a *proper* supermodel of M$'$ iff in addition, $D \neq D'$

[20] See also the end of section 2.7.

[21] Never mind whether to classify this as semantic or pragmatic.

[22] These are corrected/improved versions of the original definitions; for the main improvements I thank Tim Williamson. A model is a pair $\langle D, I\rangle$, D the domain, I the interpretation function (which assigns semantic values to nonlogical expressions). For convenience, let's here dispense with function symbols in place of predicates. For any set A, A^n is the n-place Cartesian product of A with itself. The semantic value of an n-place predicate functor (e.g., #) is a function from D^n into D.

Def of expansion quantifier q *expands* quantifier q' iff every model that depicts q' has a supermodel that depicts q. q *properly expands* q' iff in addition, q' does not expand q

Suppose, then, that $\langle m, c\rangle$ properly expands $\langle m', c'\rangle$. This is unexciting if $m = m'$; this is the case where the quantifiers in m', c' are mere restrictions of those in m, c. But it is exciting if $m \neq m'$, for then m, c is not the result of dropping restrictions on the quantifiers in m', c'. We have instead the distinctive kind of expansion. The following definitions are therefore appropriate:

Def of restriction $\langle m', c'\rangle$ is a (proper) *restriction* of $\langle m, c\rangle$ iff $m = m'$ and $\langle m, c\rangle$ (properly) expands $\langle m', c'\rangle$

Def of unrestricted A quantifier is *unrestricted* iff it is not a proper restriction of any quantifier

2.5 The form of quantifier variance theses

Various theses of quantifier variance may be formulated in terms of this apparatus.

For any quantifier, q, everyone believes in the mundane kind of quantifier variance that results from quantifier restriction:

Closure under restrictions If some supermodel of M depicts q, then there exists a restriction of q that M depicts

The quantifier variantist wants to go further. Let $\mathscr{M}$ be a set of models. Think of the models in $\mathscr{M}$ as "quantifier worlds"—models that describe what the world would be like given various quantifier meanings. Let E be a set of meanings, and let Q(E) be the set of quantifiers "based on" E (i.e., $\{\langle m, c\rangle : c$ belongs to m and $m \in \mathrm{E}\}$); the members of Q(E) will be the multiple candidate quantifier meanings in which the quantifier variantist believes. Claims of quantifier variance are then plentitude theses for Q(E); any such claim will say roughly that each quantifier world in $\mathscr{M}$ depicts some quantifier in Q(E).

This idea needs to be made precise along two dimensions. First, an appropriate *range* of quantifier worlds—i.e., what goes into $\mathscr{M}$—must be specified. That is the task of the next section. Second, the *form* of the correspondence between quantifier worlds and quantifiers must be specified. Let us turn to this second task.

The weakest form is simply this:

Weak $\mathscr{M}$/E-quantifier variance Every member of $\mathscr{M}$ depicts some member of Q(E)

But that is too weak, since it is consistent with all the quantifiers in Q(E) being restrictions on a single maximal quantifier. Say that model M *outruns* meaning m iff for no c does M depict $\langle m, c\rangle$; here are some stronger forms:[23]

Moderate $\mathscr{M}$/E-quantifier variance Weak $\mathscr{M}$/E-quantifier variance + some member of $\mathscr{M}$ outruns some member of E

Strong $\mathscr{M}$/E-quantifier variance Weak $\mathscr{M}$/E-quantifier variance + every M $\in \mathscr{M}$ outruns some member of E (provided M is a proper supermodel of some member of $\mathscr{M}$)

Unrestricted $\mathscr{M}$/E-quantifier variance Every member of $\mathscr{M}$ depicts some unrestricted member of Q(E)

Moderate quantifier variance adds the claim that at least one quantifier world is "beyond the reach" of at least one meaning—the meaning cannot be unrestricted to generate the world. Strong quantifier variance goes further by claiming that *each* quantifier world is beyond the reach of some meaning (except when the world is a "minimal" member of $\mathscr{M}$). Unrestricted quantifier variance goes the furthest: it says that each quantifier world depicts some unrestricted quantifier. Life would be simpler if we could focus solely on unrestricted quantifier variance, but I want to leave open the possibility that some meanings do not have contexts in which the quantifiers are absolutely unrestricted.

2.6 The extent of quantifier variance

Section 2.5 provided various forms of quantifier variance. These may be given content in various ways.

Each form of quantifier variance assumes given a class $\mathscr{M}$ of "quantifier-worlds" for which corresponding quantifier meanings are alleged to exist. Here is one fairly strong constraint that one might want to put on $\mathscr{M}$.[24]

Upward closure Any supermodel of a member of $\mathscr{M}$ is itself a member of $\mathscr{M}$

[23] Thanks to Joshua Brown for the moderate formulation.

[24] If one does impose this constraint then $\mathscr{M}$ can no longer be a set. It could instead be a proper class (or talk of it could be understood in terms of plural quantification.)

(Weaker versions would require only closure under certain sorts of supermodels.)

The quantifier variantist may well *not* want to impose the following inverse constraint on the set, E, of meanings:

> **Downward E-closure** If M is a supermodel of M′ and M depicts some member of Q(E), then M′ outruns some member of E

Downward closure says that we can choose arbitrarily "small" quantifiers—and not just by restriction. That is, if we're speaking one language, whose quantifier is depicted by some model, and we choose a submodel of that model, then there is some other language we can speak in which, no matter how far we unrestrict the quantifiers, we will not reach the chosen submodel. Why might our quantifier variantist not accept downward closure? Perhaps some sentences are *atomic*, in that no meaning treats them as false except because of quantifier restriction. 'There exist electrons' might be an example. The relevant notion of an atom is this:

> **Def of E-atom** Sentence ϕ is *E-atomic* iff for every $m \in$ E, there exists a c such that i) ϕ is true$^{c}_{m}$, and ii) for any c', if ϕ is not true$^{c'}_{m}$ then $\langle m, c' \rangle$ is a restriction of $\langle m, c \rangle$

Supermodels are allowed to "expand" nonlogical expressions in two ways: they can interpret new nonlogical expressions that are not interpreted by the submodel, and they can expand the extensions of nonlogical expressions that are interpreted by the submodel. Either of these degrees of freedom could be constrained, by allowing only nonlogical expressions in certain chosen sets (K and L below) to be thus expanded:

> **Def of ⟨K,L⟩-supermodel** M is a (proper) ⟨K,L⟩-supermodel of M′ iff i) M is a (proper) supermodel of M′, ii) any nonlogical expressions that are *newly* interpreted (i.e., interpreted by M but not M′) are in K, and iii) any nonlogical expressions that are *altered* (i.e., have different extensions in M and M′) are in L

There result, then, corresponding notions of ⟨K,L⟩-restriction, various forms of ⟨K,L⟩-quantifier variance, and so on.

For instance, one might allow expansions of quantifiers to be accompanied by the introduction of new nonlogical expressions for the features of "the newly introduced entities", while requiring that old nonlogical expressions have, as it were, exactly the same extensions that they originally had. The neoFregean, for instance, must add a new nonlogical expression—the predicate functor '#'—but need not change old nonlogical expressions. NeoFregeans, therefore, might restrict their quantifier variance claims to $\langle\{\text{`\#'}\},\varnothing\rangle$. NeoCarnapians, on the other hand, want the extension of 'part of' (and many other predicates) to expand when the domain is expanded, but may not need new nonlogical expressions. NeoCarnapian quantifier variance claims may therefore have the form $\langle\varnothing,\{\text{`part of'},\text{`material object'},\dots\}\rangle$.

We now have the means to formulate quantifier variance theses. As an example, one might state a form of neoCarnapian quantifier variance thus:

NeoCarnapian quantifier variance restated There is a nonempty class of models, $\mathscr{M}$, and a class of meanings, E, such that:

i) $\mathscr{M}$ obeys upward ⟨∅,{'part of', 'material object' …}⟩-closure
ii) every member of E is as natural as every other, and no meaning not in E is as natural as any meaning in E
iii) strong $\mathscr{M}$/E-⟨∅,{'part of', 'material object', …}⟩-quantifier variance is true

2.7 What else must vary?

Suppose we vary what the quantifiers mean. The quantifier variantist should, I think, say that we then also vary the meaning of every other expression distinctive of predicate logic: names, predicates, function symbols.[25] Indeed, the meanings of these *categories*, construed as semantic categories, must vary.

This can be approached first by examining the following argument against quantifier variantism. Consider two putative languages in which the quantifiers

[25] Plausibility argument: pretend that giving meaning to a language is just a matter of describing its intended model. Models are described using quantifiers in the metalanguage. One uses metalanguage quantifiers to specify a domain, which fixes the meaning of the object-language's quantifiers; and one uses metalanguage quantifiers to give the meanings of object-language constants and predicates (a constant means an object in the domain; a predicate means a set of tuples from the domain). So if one then changes the meanings of the metalanguage quantifiers, different meanings for all the object-language's expressions would ensue.

mean different things. Surely, if these languages exist, one could introduce a third language containing symbols $\exists_1$ and $\exists_2$ for the quantifier-meanings of the first two languages. But if $\exists_1$ and $\exists_2$ obey the usual inference rules then they will be provably equivalent. (E.g., suppose $\exists_1 x\phi(x)$. By $\exists_1$-elimination, ϕ(a). By $\exists_2$-introduction, $\exists_2 x\phi(x)$.)[26]

The defender of quantifier variance ought to reply that one cannot introduce a language with both $\exists_1$ and $\exists_2$ but with a common stock of names, predicates, and function symbols. For the notions of name, predicate, function symbol, and quantifier are all connected. If '$\exists_1$' is a quantifier in one sense—a quantifier$_1$—then it is only names$_1$, predicates$_1$, and function symbols$_1$ that connect to it in the usual ways. And expressions that inferentially connect to quantifiers$_2$ are not names$_1$, etc.; they are names$_2$, etc.[27]

A second route to the same conclusion emerges from reflection on a recent challenge to quantifier variantism presented by Matti Eklund (2007) and John Hawthorne (2006). Quantifier variantism allows the following scenario involving two characters, Big and Small. Big speaks a language (Biglish) in which '$\exists x$ Table(x)' is true, and introduces a name, 'a', for a table. Small, on the other hand, speaks a "smaller" language, in which one cannot quantify over tables. But Small is a quantifier variantist, and thinks that he does not genuinely disagree with Big. So even though Small does not himself accept the sentence 'Table(a)', he thinks that it is true in Biglish. But this commits Small to rejecting familiar Tarskian ideas about semantics. According to Tarskian semantics, for any language, L, a subject-predicate sentence is true-in-L iff the denotation-in-L of its subject term is a member of the extension-in-L of its predicate. If Small accepts this biconditional, then in order to admit the truth of 'Table(a)' in Biglish, Small himself would have to admit that there exists something that 'a' denotes-in-Biglish. (The quantifier 'the' in the biconditional is Small's, notice.) But there seems to be no such object—speaking *Small's* language, that is, one cannot say that such an object exists. So runs the Eklund-Hawthorne argument.

The quantifier variantist should reply as before: names and quantifiers are

[26]Compare Harris (1982); Hart (1982); Williamson (1987/8).

[27]A purely syntactic, inferentially inert, notion of grammatical category would classify names$_1$ and names$_2$ together, but would not rescue the argument.

An alternate route to blocking the argument, due to Jason Turner (2008), deserves mention. Turner claims that $\exists_1$ and $\exists_2$ obey only free-logical introduction and elimination rules. The move from ϕ(a) to $\exists_2 x\phi(x)$ would then be invalid because $\exists_2$-introduction would require the additional premise $\exists_2 x\, x$=a.

connected. Small should deny that Big's expression 'a' is a name (i.e., deny that it is a name$_{\text{Small}}$.)[28]

This reply is, I think, *correct*, but it doesn't fully answer Eklund and Hawthorne. For even if Small is right to deny that Biglish contains names or subject-predicate sentences, it would be hard for Small to deny that Big's use of language is in *some* sense compositional. And so, shouldn't Small say something systematic about how Big's sentences get their truth conditions?

Yes; but Small need not stick to the book in doing so. *Anyone* can agree that *some* extreme cases call for novel semantic ideas in order to make sense of alien but compositional linguistic behavior. From Small's point of view, the case of Big calls for a (slight) departure from the Tarskian paradigm: Big's sentence $\ulcorner F\alpha \urcorner$ is true, Small might say, iff there are some referent*s* (plural) of the "subject" term α that are in the "extension" (in a plural sense) of the "predicate" F. The resulting theory might be complex and ugly. But if a full semantics is difficult (or even impossible) to give using Small's language, that wouldn't undermine quantifier variance. Granted, it would be an asymmetry between Small and Big, for there is no corresponding disadvantage to speaking Biglish. But quantifier variantists can admit that bigger is better for certain purposes; all they are committed to saying is that neither language adheres better to nature's joints. (French may be the language of love, but is no better for it ontologically speaking.)

3. The epistemic goal of neoFregeanism

My reading of neoFregeanism appeals to quantifier variance. In essence: the neoFregean's claim that states of affairs can be "reconceptualized" as involving quantification over abstracta is a metaphor for the claim that i) there is a meaning for the quantifiers on which one can quantify over abstracta, and ii) this meaning is not a "second-class" citizen: it is just as natural as quantifier-meanings on which one cannot so quantify. In order to evaluate whether this view is an adequate reading *of neoFregeanism*, we must ask what neoFregeans want out of their theory, epistemically speaking.

Suppose you begin life as a platonist. You are convinced that there are many abstract entities, including numbers. In that case, you should be happy to accept Hume's principle.[29] Where there are finitely many Fs, you would take

[28] Compare Hirsch (2002*b*, p. 57).

[29] Setting aside the issues of note 8.

'the number of Fs' to pick out the appropriate one of these abstract entities that you antecedently accept. Indeed, if you heard a neoFregean saying that Hume's Principle is a "definition", you might simply take that as information about which of the functional correlations between pluralities and objects that you antecedently believe in, is to be associated with 'the number of'.

NeoFregeans want more than that. Their definition is supposed to have an epistemic payoff. You are not supposed to need an antecedent commitment to abstracta in order to accept Hume's Principle and subsequently derive Arithmetic. The neoFregean program is supposed to erase doubts about abstracta. But how?

In the remainder of this section I want to do a few things. Ultimately I want to suggest that the potential epistemic payoff of neoFregeanism is more modest than what is usually supposed. I want thereby to dispel the false impression that neoFregeanism provides a way to avoid substantive metaphysical questions about mathematical existence. And I hope to clarify questions about the status of the logical knowledge that neoFregeans must presuppose.

3.1 No detour around substantive metaphysics

Platonists seem to face an epistemic problem. If mathematics is about a realm of mind-independent abstract entities, then how do we know about these entities? Models of other sorts of knowledge—perceptual, testimonial, historical, and scientific knowledge, for example—do not seem to apply to mathematics.

A powerful motivation for neoFregeanism is that it promises to solve this epistemic problem. The problem, one might think, is created by the traditional approach to ontology (the "philosophical science of ontology" that Wright (1983, p. 52) deplores).

In fact this motivation is illusory. In effect, what neoFregeans are trying to do is argue for an underlying metaontology[30] (theory of the nature of ontology) that guarantees the success of their stipulation of Hume's Principle. They hope thereby to dispel doubts about mathematics. But in order to dispel all doubts, it is not enough that the underlying metaontology be *true*. It must itself be epistemically secure. And models of perceptual knowledge, testimonial knowledge, and the like are of no more help in understanding how we could know the truth of neoFregean metaontology—a substantive bit of metaphysics—than they are in understanding how we could have mathematical knowledge.

[30]The term is from van Inwagen (1998).

Consider, for instance, the quantifier variance interpretation of neoFregeanism. If an appropriate quantifier variance hypothesis is true (section 5), then the stipulation of Hume's Principle is bound to succeed. But quantifier variance itself is a substantive metaphysical hypothesis. An alternate hypothesis is that there is a single most natural quantificational meaning—a distinguished quantifier. Call this view *ontological realism*. Never mind whether it is true; what is important is that neoFregeanism, on the quantifier variance interpretation, is committed to its falsity. The rejection of ontological realism in favor of quantifier variance is, if anything, less epistemically secure than the mathematical knowledge it is supposed to ground.

The point is not limited to the quantifier variance interpretation. On any interpretation, neoFregeanism will be committed to the falsity of rival metaontological positions. Far from providing a detour around substantive fundamental metaphysics, neoFregeanism is itself a piece of substantive fundamental metaphysics.

3.2 A modest goal

None of this counts against neoFregeanism. On the contrary, it should be liberating. Once the goal of dispelling all arithmetic doubt by avoiding substantive metaphysics is off the table, neoFregeans can set themselves a more attainable goal: improving our epistemic position.

NeoFregeans need a metaontological hypothesis that guarantees the success of the stipulation of Hume's Principle. On the quantifier variance interpretation, as well as on another interpretation I will discuss, the needed metaontological hypothesis has independent plausibility. Thus, showing that the hypothesis guarantees the success of the stipulation could be argued to improve our epistemic position. For one route to epistemic improvement—perhaps the best route when it comes to the most fundamental matters—is to embed less certain beliefs within an attractive, explanatory, and general theory. Showing that mathematical knowledge can be thus embedded would not dispel all doubts about mathematics, but that was never in the cards anyway. Improving our epistemic position is a modest but attainable goal.

3.3 Logical knowledge

Second-order logic is needed to derive the Peano axioms from Hume's Principle. So neoFregeans need an account of second-order logical knowledge to complete their mathematical epistemology. I want to comment briefly on two questions that are generally considered relevant here. First, is second-order logic really logic? Second, is second-order logic really set theory?

My comment about the first question is really just an opinionated remark. If we had an account of our knowledge of *first*-order logic, then it might matter whether second-order logic is logic (for it might affect whether our account of first-order logical knowledge would carry over to second-order logic.) But we don't, so it doesn't.[31]

Second question: is second-order logic just set theory in disguise, as Quine (1970) thought? If it is, then neoFregeanism provides no more secure an epistemic foundation for Arithmetic than that provided by the more usual reduction of Arithmetic to set theory plus definitions. NeoFregeans are thus committed to "innocent" second-order quantification.[32] But is such quantification possible?

As many have pointed out, the fact that the standard model theory for second-order logic is set-theoretic is neither here nor there, for the standard model theory for first-order logic is also set-theoretic, and no one thinks that first-order reasoning is implicitly set-theoretic.

Also neither here nor there is the following. Suppose platonism about set theory is true, and imagine the semantic gods looking down upon an innocent who uses second-order quantifiers and variables. The semantic gods might well interpret the innocent as quantifying over sets. In the same way, the semantic gods might interpret a pre-Einsteinian innocent as meaning by 'simultaneous', simultaneity-in-her-own-frame-of-reference; or a pre-Parsonian as quantifying over events when saying 'I walked quickly down the street'.[33] Thus, since platonism may well be true, second-order quantification may well "semantically commit" one to sets. This is neither here nor there because the question is whether it can be *established* in the current dialectical context that one cannot use higher-order quantification without believing in sets.

So how should we approach the question of whether innocent second-order quantification is possible? The following move sharpens the debate. Let our neoFregean *stipulate* that her second-order quantifiers are to be understood

[31] What is behind this opinionated remark is opposition to leading attempts to explain first-order logical knowledge: logical conventionalism (against which see Quine (1936); Sider (MS)) and the view that logical knowledge is fully explained by linguistic knowledge (against which see Prior (1960); Horwich (1997); Williamson (2003); Field (2006)).

[32] See, for instance, Rayo and Yablo (2001).

[33] Parsons (1990).

innocently, as not quantifying over sets. What form must opposition to innocent second-order quantification now take?

It must turn into the charge that the second-order quantifiers are semantically defective by virtue of underspecification. The second-orderist's usage of the allegedly innocent quantifiers settles *some* things about how they are to behave. For instance, the second-orderist's usage might settle that every sentence of the form $\ulcorner \mathrm{Ga} \urcorner$, with G a predicate constant, is to imply $\ulcorner \exists F F \mathrm{a} \urcorner$. But, the criticism would be, when we reach second-order quantifications whose truth values are not settled by the actual usage of second-orderists, then everything goes fuzzy: there are no determinate truth values. In essence, if there are no sets, then no other Wittgensteinian "rails to infinity" are available to supply semantic determinacy in cases that are not settled by usage.[34]

Conversely, since the neoFregean who stipulates an innocent usage of second-order quantifiers is committed to the semantic determinacy of her language, she is committed to there being sufficient structure in the world to provide the Wittgensteinian rails.[35]

4. The maximalist interpretation

Before discussing the quantifier variance interpretation of neoFregeanism further, I want to distinguish it from another interpretation: the "maximalist" interpretation.[36] The distinction can be brought out by asking the question, if Hume's principle is to be a definition, then what expression or expressions does it define? As we'll see, the quantifier variantist thinks that it defines the quantifiers. The maximalist denies this.[37] According to the maximalist, in laying down Hume's Principle as a definition, we keep the quantifiers meaning exactly what they did before, and stipulate that '#' is to be interpreted so that Hume's Principle comes out true.

In that case, one might ask, how could we be certain that the implicit definition succeeds? Mightn't the requisite objects be missing?

Well, suppose it's just a fact about the nature of existence that, in a sense to be explored, anything that can exist, does exist. That is, existence is quite

[34] Wittgenstein (1958, §218).

[35] The question of how exactly to articulate this commitment to structure is a difficult one; but there is no question that there is indeed such a commitment. See Sider (2009).

[36] Here I am indebted to Eklund (MS, 2006*b*). The term 'maximalist' is his.

[37] And therefore has no need for quantifier variance. The maximalist could, for instance, be an ontological realist.

generally maximal—maximalism. Then if Hume's Principle is consistent, there must be objects satisfying Hume's Principle.[38]

Of course, the *truth* of maximalism wouldn't on its own dispel all ontological doubts about arithmetic, for one could doubt that maximalism is true. But as I explained in section 3.2, modest epistemic progress would be made if an attractive general hypothesis about metaontology were found on which stipulations like Hume's Principle invariably succeed.

And—perhaps contrary to appearances—maximalism is indeed a reasonably attractive hypothesis. Maximalism is tempting (to the degree that it is) because it minimizes arbitrariness. If maximalism is false, and some consistent objects are present while others are missing, there's a why-question without an answer: why do these objects, but not those, exist? Whereas if maximalism is true, we have a nicely rounded picture of the world, and fewer why-questions go unanswered. Maximalism is attractive for the same reason that plenitudinous views about material ontology are attractive.

The more general the maximalism, the more it minimizes arbitrariness. For instance, maximalism might be extended beyond the realm of the abstract into the realm of the concrete: temporally (B-theory, perdurance), modally (modal realism), and/or existentially (Meinong). Of course, this may be taking things too far—there's more to epistemic life than minimizing arbitrariness. (For my money, modal, existential, and abstract maximalisms go too far, but I'm not going to try to evaluate maximalism here; my point is just that it has its charms.)

"Everything that can exist, does exist"—what exactly does that amount to, even confining our attention to mathematics? Here are three unacceptable interpretations:

(M1) Every abstraction principle can be truly interpreted

(M2) Every consistent abstraction principle can be truly interpreted

(M3) Every conservative consistent abstraction principle can be truly interpreted

[38] A maximalist could bypass Hume's Principle and infer the truth of the Peano Axioms directly (though perhaps the abstraction principles are better candidates to be partial definitions of natural language number terms.) Similar remarks apply to the quantifier variance interpretation of the next section. Hume's Principle fails to be central on my interpretations of neoFregeanism because I refused in section 1.3 to view the priority of syntax as an important issue. Thanks to Matti Eklund here; and see Eklund (2006*b*, section III).

An abstraction principle is a principle of the form:

$$\forall F \forall G(\alpha x{:}Fx = \alpha x{:}Gx \leftrightarrow \phi(F,G))$$

where ϕ expresses an "equivalence relation between the concepts F and G". But some abstraction principles are contradictory. Frege's basic law V is an example. So (M1) is false.

(M2) is false because there are pairwise consistent abstraction principles that are jointly inconsistent. Let's take George Boolos's (1990) example of parity:

> **Definition** F and G *differ evenly* iff the things, x, such that $(Fx \wedge {\sim}Gx) \vee (Gx \wedge {\sim}Fx)$, are even (and finite) in number
>
> **Parity abstraction principle** $\forall F \forall G(\mathrm{P}x{:}Fx = \mathrm{P}x{:}Gx \leftrightarrow F$ and G differ evenly)

Boolos shows that the Parity abstraction principle is consistent, but is only true in finite domains. Hume's Principle is consistent, but is only true in infinite domains. So each is consistent, but they can't be true together.

In response to Boolos, Wright proposed that acceptable abstraction principles must be conservative, in a certain sense. Not the usual sense (namely, that nothing in the old vocabulary that was unprovable before the introduction of the abstraction principle becomes provable after its introduction), for Hume's Principle isn't conservative in that sense: "there are infinitely many things" is stateable in second-order logic, isn't a logical truth, but is a consequence of Hume's Principle. Wright's conservativeness requirement is rather that nothing about the extensions *of old concepts* can follow from the added abstraction principle. The parity principle implies, with respect to each primitive predicate F, that its extension must be finite, and so is not conservative in the relevant sense. However, it turns out that there are inconsistent pairs of individually conservative abstractions.[39]

So the view will have to be much more subtle. Kit Fine (2002) develops a sophisticated theory of when abstractions succeed, which could be taken over by maximalists in order to articulate the precise sense in which mathematical existence is maximal. Matti Eklund (MS) discusses the prospects of a general maximalism.

[39] See Shapiro and Weir (1999); and see MacBride (2003, pp. 145–146) for further discussion and references.

5. The quantifier variance interpretation

Return to the question: if Hume's principle is a definition, then what expressions does it define? According to the quantifier variance interpretation, the answer is that Hume's principle constrains the interpretation of the quantifiers, as well as 'the number of'. (Given section 2.7, it thereby constrains the interpretation of every other predicate logic expression.) The idea is to stipulate that the quantifiers are to be interpreted so that Hume's Principle comes out true. And an appropriate version of the doctrine of quantifier variance will guarantee that the quantifiers *can* be so interpreted.

In light of section 3, the goal is not to make the epistemology of mathematics utterly unproblematic. To erase all doubts, one would need to know that quantifier variance is true. Still, if quantifier variance has independent plausibility, the neoFregean will have integrated mathematics into a plausible general metaontology, thus making epistemic progress. And quantifier variance does indeed have independent plausibility: like maximalism, it minimizes arbitrariness.

5.1 Neofregean quantifier variance stated

On the quantifier variance interpretation, when we lay down Hume's Principle as a definition, we're no longer assuming that the principle can be rendered true under the old meaning of the quantifiers. The idea is to *change* what the quantifiers mean.

The view is *not* that, after introducing Hume's Principle, the quantifiers in mathematical sentences mean something different from the quantifiers in nonmathematical sentences. The idea is rather that all quantifiers throughout the language have changed. In the new language, one can say that mathematical entities and physical entities exist in the same sense.

But we *don't* want to say that statements about nonmathematical entities change their truth values. The meaning shift ought to be conservative, in a certain sense. Not of course in the strictest sense, for as noted above, Hume's Principle forces an infinite domain. And in fact, we can't even say quite what Wright says about the conservativeness of acceptable abstractions: that no constraints may be put on the extensions of primitive nonmathematical (and nonlogical) predicates. Since the notion of an extension is defined using quantifiers, after the quantifiers change meaning one cannot strictly speak of the old predicates as having extensions at all (they may not even be rightly called predicates.) Instead we can offer an appropriate account of quantifier variance using the apparatus of section 2. Here is a stab at it:[40]

NeoFregean quantifier variance There is a nonempty class of models, $\mathscr{M}$, and a class of meanings, E, such that:

i) $\mathscr{M}$ obeys upward $\langle\{\text{‘\#’}\},\varnothing\rangle$-closure
ii) every member of E is as natural as every other, and no meaning not in E is as natural as any meaning in E
iii) strong $\mathscr{M}$/E-$\langle\{\text{‘\#’}\},\varnothing\rangle$-quantifier variance is true

Some help in unpacking. $\mathscr{M}$ is nonempty, so it contains at least one "initial" quantifier world. Upward $\langle\{\text{‘\#’}\},\varnothing\rangle$-closure will then force it to include all $\langle\{\text{‘\#’}\},\varnothing\rangle$-supermodels of the initial world, which "add" new entities to those present in the initial world. For each of these worlds, according to the claim, there is a corresponding (i.e., depicted) quantifier. Recall that restricting a quantifier variance claim to the ordered pair $\langle K,L\rangle$ constrains what nonlogical expressions can apply to the "newly introduced objects". A restriction to $\langle\{\text{‘\#’}\},\varnothing\rangle$ means that the newly introduced entities may not enter into the extensions of nonlogical predicates occurring in the initial world; they can only be semantic values of '#'.

5.2 Which abstraction principles are acceptable?

The problem of incompatible stipulations—individually consistent but pairwise inconsistent abstraction principles—undermined reading (M2) of maximalism. The quantifier variance neoFregean has a little more latitude.

Hume's Principle is true only in infinite domains. The Parity abstraction principle is true only in finite domains. So under no one quantifier meaning can both be true. But they can be true under different quantifier meanings.

[40]How far beyond the bounds of this thesis does quantifier variance extend? As Joshua Brown pointed out to me, quantifier variantists face hard questions here. Is there, for example, a maximally natural quantifier meaning on which nothing at all exists? Is there one on which gods exist? There is a continuum of available positions here. At one end, maximally natural quantifier meanings proliferate; the more alien-seeming ones are merely semantically deficient; their only sin is that they do not fit our actual *use* of quantificational language. On the other, there are fewer maximally natural candidate meanings, perhaps only those guaranteed by the thesis stated in the text. The first end of the continuum best minimizes arbitrariness, but at a terrible cost: surely *some* ontological questions (e.g., 'are there gods?', 'are there extra-terrestrials?') have "objective" answers!

Thus, more abstraction principles are available to the quantifier variantist, so long as they are not all introduced at the same time.

Quantifier variantists could investigate the conditions under which abstractions succeed—that is, the conditions under which there exists a quantifier meaning on which a given abstraction principle comes out true. The following simple theorem is a start:[41]

> **Success** Suppose neoFregean quantifier variance is true. Suppose a certain abstraction principle A, in which the only nonlogical expression is '#', is consistent. Then there is some maximally natural meaning m and context c such that A is true^c_m

5.3 Give the people what they want

The quantifier variance interpretation gives neoFregeans all they can reasonably hope for, if not absolutely everything they want. It lets them say that abstract objects exist and are mind-independent, while claiming that in some sense the definitions that "introduce" them are bound to succeed.

NeoFregean quantifier variance is an underlying metaontology on which consistent abstraction principles can invariably be truly interpreted. These abstraction principles may be put forward as implicit definitions of mathematical expressions plus the apparatus of predicate logic (quantifiers, names, predicates, function symbols). The resulting quantificational language will be at least as good as any other quantificational language, and the propositions expressed in the new language will be perfectly mind-independent.

Individually consistent but pairwise inconsistent abstractions can be truly interpreted on different interpretations of the quantifiers, even though they cannot be simultaneously truly interpreted.

The quantifier variance interpretation makes sense of the neoFregean idea that quantification is "light", not a big deal. If there were a single distinguished quantificational meaning, then it would be an open possibility that numbers,

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[41] Proof: understand talk of supermodels, etc., as relativized to $\langle\{\text{`\#'}\},\varnothing\rangle$ throughout. Let M_0 be a member of $\mathscr{M}$. Let M be a model of A (choose M so that its domain does not overlap that of M_0.) Construct model M^+ by combining the domains of M and M_0, and keeping the extensions of all nonlogical expressions fixed. M^+ is a supermodel of M_0; thus, by upward closure, $\mathrm{M}^+ \in \mathscr{M}$, and so by strong $\mathscr{M}/\mathrm{E}$ quantifier variance, M^+ depicts $\langle m, c'\rangle$ for some (maximally natural) $m \in \mathrm{E}$ and context c'. Now, M^+ is a supermodel of M as well; so by closure under restrictions, M depicts $\langle m, c\rangle$ for some c. A is true in M, and so is true^c_m.

directions, and other abstracta are simply *missing* from existence in the distinguished sense of 'existence', even though we speak in a perfectly consistent way about them. We would have to approach the question of whether they exist by some means other than assessing the consistency of their postulation—we would need the dreaded "philosophical science of ontology". That's heavy. But if quantifier variance is true, then this is not an open possibility.

Regarding the epistemology of arithmetic, once you spot yourself the truth of neoFregean quantifier variance (and spot yourself second-order consequence!), the introduction of numbers becomes relatively epistemically unproblematic. Now, I see no hope of establishing neoFregean quantifier variance itself beyond a shadow of a doubt. But that thesis is an attractive general thesis about metaontology. Integration of mathematics into an attractive and general theory is the most epistemic progress we can hope for.

Most importantly, the quantifier variance interpretation is a way—the *only* way, as far as I can see—of making sense of the idea that abstraction principles "reconceptualize" facts about, e.g., parallelism and equinumerosity. The core of quantifier variance is that the facts do not demand a unique description in the language of quantifiers. The facts about parallelism can be described by saying "there are only lines", or they can be described by saying "there are lines and directions". Just as one can describe the facts of distance using any chosen unit of measure, one can describe the facts of ontology using any chosen quantificational meaning.

And the content of the quantifier variance interpretation is clearer than the intriguing but elusive texts of real live neoFregeans. That's not to say that it is *true*. I myself reject it, because I reject quantifier variance.[42] But I prefer an enemy that I can understand.

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[42] See Sider (2009).

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