Supplementary exercises for *Logic for Philosophy*

Semantics of propositional logic

Show that:

1. $\vDash_{\text{PL}} ((P \vee Q) \rightarrow R) \rightarrow (P \rightarrow R)$
2. $\vDash_{\text{PL}} ((P \wedge Q) \rightarrow R) \rightarrow ((Q \wedge {\sim}R) \rightarrow {\sim}P)$
3. $\vDash_{\text{PL}} (P \leftrightarrow (Q \leftrightarrow R)) \rightarrow ((P \leftrightarrow Q) \leftrightarrow R)$
4. $\nvDash_{\text{PL}} ((P \wedge Q) \rightarrow R) \rightarrow (Q \rightarrow R)$
5. $\nvDash_{\text{PL}} ((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (Q \wedge R)$
6. $P \vee Q, P \rightarrow R, Q \rightarrow R \vDash_{\text{PL}} R$
7. $P \rightarrow Q, Q \rightarrow R, R \rightarrow {\sim}P \vDash_{\text{PL}} {\sim}P$
8. $P \rightarrow (Q \vee R), Q \wedge R \nvDash_{\text{PL}} {\sim}P$
9. $P \leftrightarrow Q$ and $(P \wedge Q) \vee ({\sim}P \wedge {\sim}Q)$ are semantically equivalent (in PL)
10. ${\sim}(P \wedge Q)$ and ${\sim}P \wedge {\sim}Q$ are not semantically equivalent (in PL)

Axiomatic proofs in propositional logic

Before toolkit:

1. Show that $P{\to}(P{\to}Q) \vdash (P{\to}Q)$ (you may use exercises 2.4)

After toolkit:

2. Show that $\phi{\to}\psi, \chi{\to}\psi, {\sim}\phi{\to}\chi \vdash \psi$.
   (Given the definition of "$\vee$", this is reasoning by cases: $\phi{\to}\psi, \chi{\to}\psi, \phi\vee\chi \vdash \psi$.)
3. Show that if $\Gamma, \phi \vdash \psi$ and $\Delta, \phi \vdash {\sim}\psi$ then $\Gamma, \Delta \vdash {\sim}\phi$ (this is the principle of reductio ad absurdum).
4. Show that $\vdash ((\phi{\to}\psi){\to}\phi) \to \phi$ ("Peirce's Law")

Metalogic for PL

Some practice with inductive proofs:

1. Show that every wff has at least one sentence letter.

2. Show that every wff has twice as many parentheses as $\rightarrow$s.

3. Show that in the interpretation in which every sentence letter is true, every formula with no $\sim$s is true.

After the completeness theorem section:

4. (Long) The "strong completeness theorem" says the following: *for any set of wffs* Γ *and any wff* ϕ, *if* $\Gamma \vDash \phi$ *then* $\Gamma \vdash \phi$. Prove the strong completeness theorem.

   Hint: pattern your proof after the proof of the regular completeness theorem from section 2.9.4. Just as in section 2.9.4, the lemmas and theorems from sections 2.8 and 2.9 will be needed.

5. The "compactness theorem" for propositional logic says that if a set semantically implies a formula, then some finite subset of that set also semantically implies the formula; i.e.,: *For any set of wffs* Γ *and wff* ϕ, *if* $\Gamma \vDash \phi$ *then for some finite* $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vDash \phi$. Prove the compactness theorem.

   Hint: the following items will be useful: i) strong soundness (exercise 2.9 from the book), ii) one of the other exercises on this page, and iii) one of the theorems or lemmas from section 2.9.

Three-valued logic

1. In section 3.4.3 we introduced a new symbol, $\triangle$. But actually (as Tarski showed), in Łukasiewicz's system, $\triangle$ can be *defined* in terms of $\sim$ and $\rightarrow$, as follows:

   "$\triangle\phi$" is short for "$\sim(\phi\rightarrow\sim\phi)$"

   (a) Show that given Łukasiewicz's tables for $\sim$ and $\rightarrow$, this definition does indeed generate the right truth table for $\triangle$, i.e.:

$\triangle$	
1	1
0	0
#	0

   (b) Show that the definition does *not* generate this truth table given Kleene's tables for $\sim$ and $\rightarrow$.

2. Call the "semantic deduction theorem" the statement that if $\phi \vDash \psi$ then $\vDash \phi\rightarrow\psi$. (Here ϕ and ψ may be any wffs given our original definition of a wff from chapter 3. Thus the *primitive* connectives in ϕ and ψ can only be $\sim, \rightarrow, \wedge, \vee$, and $\leftrightarrow$, although the result of starting with $\triangle$ and then replacing it with Tarski's definition is of course allowed.) Does the semantic deduction theorem hold for…

   (a) …Łukasiewicz's system? (Hint: consider Tarski's defined $\triangle$.)

   (b) …Kleene's system?

   (c) …the logic of paradox?

   Does the converse of the semantic deduction theorem (i.e., "if $\vDash \phi\rightarrow\psi$ then $\phi \vDash \psi$") hold for…

   (d) …Łukasiewicz's system?

   (e) …Kleene's?

   (f) …the logic of paradox? (Hint: consider ex falso quodlibet.)

3. Write out an "official" definition of the Kleene valuation function KV (as we did for the Łukasiewicz valuation function ŁV on pp. 75–6).

Validity and invalidity in modal logic

For each of the following wffs, give a countermodel for every system in which it is not valid, and give a semantic validity proof for every system in which it is valid. When you use a single countermodel or validity proof for multiple systems, indicate which systems it is good for.

1. $\Diamond\Box(P\rightarrow\Box\Diamond P)$
2. $\Box(\Diamond P\rightarrow Q)\leftrightarrow\Box(P\rightarrow\Box Q)$
3. $\Box(P\rightarrow\Diamond Q)\rightarrow(\Diamond P\rightarrow\Diamond Q)$

Quantified modal logic

Give a validity proof if the wff is SQML-valid, and a countermodel if it is invalid.

1. $\Box\forall x(Fx\rightarrow Gx)\rightarrow\forall x(Fx\rightarrow\Box Gx)$
2. $\Box\forall x(Fx\rightarrow\Box Gx)\rightarrow(Fa\rightarrow Ga)$