Modal Propositional Logic Theorems

Note: Here are some proofs of selected problems from the theorem sheet. I'll make use of some of the shortcuts we've discussed in class.

K-Theorems

3. $\Diamond(P\&Q) \rightarrow (\Diamond P\&\Diamond Q)$

1.	$(P\&Q) \rightarrow P$	PL
2.	$\Box[(P\&Q) \rightarrow P]$	1, NEC
3.	$\Box[(P\&Q) \rightarrow P] \rightarrow [\Diamond(P\&Q) \rightarrow \Diamond P]$	K$\Diamond$
4.	$\Diamond(P\&Q) \rightarrow \Diamond P$	2,3 MP
5.	$\Diamond(P\&Q) \rightarrow \Diamond Q$	Similar to 1-4
6.	$\Diamond(P\&Q) \rightarrow (\Diamond P\&\Diamond Q)$	4, 5, PL

8. $\sim\Diamond(Q\&R) \leftrightarrow \Box(Q \rightarrow \sim R)$

1.	$\sim(Q\&R) \rightarrow (Q \rightarrow \sim R)$	PL
2.	$\Box[\sim(Q\&R) \rightarrow (Q \rightarrow \sim R)]$	1, NEC
3.	$\Box\sim(Q\&R) \rightarrow \Box(Q \rightarrow \sim R)$	2, K
4.	$\sim\Diamond(Q\&R) \rightarrow \Box\sim(Q\&R)$	MN
5.	$\sim\Diamond(Q\&R) \rightarrow \Box(Q \rightarrow \sim R)$	3, 4 PL (done left to right; now do right-to-left)
6.	$(Q \rightarrow \sim R) \rightarrow \sim(Q\&R)$	PL
7.	$\Box(Q \rightarrow \sim R) \rightarrow \Box\sim(Q\&R)$	5, NEC, K$\Box$
8.	$\Box\sim(Q\&R) \rightarrow \sim\Diamond(Q\&R)$	MN
9.	$\Box(Q \rightarrow \sim R) \rightarrow \sim\Diamond(Q\&R)$	7, 8, PL
10.	$\sim\Diamond(Q\&R) \leftrightarrow \Box(Q \rightarrow \sim R)$	9, 5, PL

21. $\Diamond(P \rightarrow Q) \leftrightarrow (\Box P \rightarrow \Diamond Q)$

(This one's a bit tough, the trick is getting the right order for your PL tautologies, in the first half, and getting the right PL strategy for the second half.)

1.	$P \rightarrow [(P \rightarrow Q) \rightarrow Q]$	PL
2.	$\Box P \rightarrow \Box[(P \rightarrow Q) \rightarrow Q]$	1, NEC, K
3.	$\Box[(P \rightarrow Q) \rightarrow Q] \rightarrow [\Diamond(P \rightarrow Q) \rightarrow \Diamond Q]$	K$\Diamond$
4.	$\Box P \rightarrow [\Diamond(P \rightarrow Q) \rightarrow \Diamond Q]$	2, 3, PL

5.	$\Diamond(P\rightarrow Q) \rightarrow (\Box P\rightarrow\Diamond Q)$	4, PL

Time out for a moment. To go the other direction, we need the right PL strategy. What we're looking for is $(\Box P\rightarrow\Diamond Q)\rightarrow\Diamond(P\rightarrow Q)$. Its contrapositive is $\sim\Diamond(P\rightarrow Q) \rightarrow \sim(\Box P\rightarrow\Diamond Q)$. Since I can get $\sim\Diamond(P\rightarrow Q)$ from $\Box\sim(P\rightarrow Q)$ (by modal negation), my intermediate goal will be to get $\Box\sim(P\rightarrow Q)\rightarrow\sim(\Box P\rightarrow\Diamond Q)$. Now, how will I get that? The consequent is equivalent in PL to $\Box P\&\sim\Diamond Q$. And in PL, you can get something of the form $\phi\rightarrow(\psi\&\chi)$ from two conditionals: $\phi\rightarrow\psi$ and $\phi\rightarrow\chi$. So: my two first tasks will be to establish $\Box\sim(P\rightarrow Q)\rightarrow\Box P$ and $\Box\sim(P\rightarrow Q)\rightarrow\sim\Diamond Q$. OK, back to the proof:

6.	$\sim(P\rightarrow Q)\rightarrow P$	PL	
7.	$\Box\sim(P\rightarrow Q)\rightarrow\Box P$	6, NEC, K$\Box$	(done task 1)
8.	$\sim(P\rightarrow Q)\rightarrow\sim Q$	PL	
9.	$\Box\sim(P\rightarrow Q)\rightarrow\Box\sim Q$	8, NEC, K$\Box$	
10.	$\Box\sim Q\rightarrow\sim\Diamond Q$	MN	
11.	$\Box\sim(P\rightarrow Q)\rightarrow\sim\Diamond Q$	9, 10, PL	(done task 2)
12.	$\Box\sim(P\rightarrow Q)\rightarrow\sim(\Box P\rightarrow\Diamond Q)$	7, 11, PL	(done intermediate goal)
13.	$(\Box P\rightarrow\Diamond Q)\rightarrow\sim\Box\sim(P\rightarrow Q)$	12, PL	
14.	$(\Box P\rightarrow\Diamond Q)\rightarrow\Diamond(P\rightarrow Q)$	definition of $\Diamond$	
15.	$\Diamond(P\rightarrow Q) \leftrightarrow (\Box P\rightarrow\Diamond Q)$	14, 5, PL	

22. $\Diamond P \rightarrow (\Box Q\rightarrow\Diamond Q)$

1.	$Q\rightarrow(P\rightarrow Q)$	PL
2.	$\Box Q\rightarrow\Box(P\rightarrow Q)$	1, NEC, K
3.	$\Box(P\rightarrow Q) \rightarrow (\Diamond P\rightarrow\Diamond Q)$	K$\Diamond$
4.	$\Diamond P \rightarrow (\Box Q\rightarrow\Diamond Q)$	2, 3, PL

24. $[\Box\Diamond P\&\Diamond\Box(P\rightarrow Q)] \rightarrow \Diamond\Diamond Q$

1.	$\Box(P\rightarrow Q) \rightarrow (\Diamond P\rightarrow\Diamond Q)$	K$\Diamond$
2.	$\Diamond P \rightarrow [\Box(P\rightarrow Q)\rightarrow\Diamond Q]$	1, PL
3.	$\Box\Diamond P \rightarrow \Box[\Box(P\rightarrow Q)\rightarrow\Diamond Q]$	2, NEC, K
4.	$\Box[\Box(P\rightarrow Q)\rightarrow\Diamond Q] \rightarrow [\Diamond\Box(P\rightarrow Q)\rightarrow\Diamond\Diamond Q]$	K$\Diamond$
5.	$[\Box\Diamond P\&\Diamond\Box(P\rightarrow Q)] \rightarrow \Diamond\Diamond Q$	3, 4, PL

D-Theorems

26. $\Box\Box P\rightarrow\Diamond\Diamond P$

1.	$\Box P \rightarrow \Diamond P$	D
2.	$\Box(\Box P \rightarrow \Diamond P)$	1, Nec
3.	$\Diamond\Box P \rightarrow \Diamond\Diamond P$	2, K◊
4.	$\Box\Box P \rightarrow \Diamond\Box P$	D
5.	$\Box\Box P \rightarrow \Diamond\Diamond P$	3, 4, PL

32. $\Diamond\Box(P\&Q)\rightarrow\Diamond\Diamond P$

My intermediate goal is to establish $\Box(P\&Q)\rightarrow\Diamond P$. Then I'll add diamonds to both sides using K◊:

1.	$(P\&Q)\rightarrow P$	PL
2.	$\Box(P\&Q)\rightarrow\Box P$	1, NEC, K□
3.	$\Box P\rightarrow\Diamond P$	D
4.	$\Box(P\&Q)\rightarrow\Diamond P$	2, 3, PL (intermediate goal reached)
5.	$\Box[\Box(P\&Q)\rightarrow\Diamond P]$	4, Nec
6.	$\Box[\Box(P\&Q)\rightarrow\Diamond P]\rightarrow(\Diamond\Box(P\&Q)\rightarrow\Diamond\Diamond P)$	K◊
7.	$\Diamond\Box(P\&Q)\rightarrow\Diamond\Diamond P$	5, 6 MP

T-Theorems

37. $\Diamond(P\rightarrow\Box Q) \rightarrow (\Box P\rightarrow\Diamond Q)$

We're trying to go from a ◊ and a □ to a ◊. That's one of our standard K problems. So in this case, our intermediate goal should be to get $P\rightarrow((P\rightarrow\Box Q)\rightarrow Q)$. (Note that the P part comes first, because it's P that is inside the □; $P\rightarrow\Box Q$ is inside the diamond.) OK, how then do we get that? Well, it follows in PL from $\Box Q\rightarrow Q$. (That's not really obvious, but if you think about it you'll see that it's true.)

1.	$\Box Q\rightarrow Q$	T
2.	$P \rightarrow [(P\rightarrow\Box Q)\rightarrow Q]$	1, PL
3.	$\Box P \rightarrow \Box[(P\rightarrow\Box Q)\rightarrow Q]$	2, NEC, K
4.	$\Box[(P\rightarrow\Box Q)\rightarrow Q] \rightarrow [\Diamond(P\rightarrow\Box Q)\rightarrow\Diamond Q]$	K◊
5.	$\Box P\rightarrow[\Diamond(P\rightarrow\Box Q)\rightarrow\Diamond Q]$	3, 4, PL
6.	$\Diamond(P\rightarrow\Box Q) \rightarrow (\Box P\rightarrow\Diamond Q)$	5, PL

S4-Theorems

43. $\Box P \rightarrow \Box\Diamond\Box P$ (Notice that this is a B theorem too, since it's an instance of B◊; #44, #47, #48, and #49 are also B theorems.)

1.	$\Box P \rightarrow \Diamond\Box P$	T◊
2.	$\Box\Box P \rightarrow \Box\Diamond\Box P$	1, NEC, K
3.	$\Box P \rightarrow \Box\Box P$	S4
4.	$\Box P \rightarrow \Box\Diamond\Box P$	2, 3, PL

47. $\Box\Diamond\Box\Diamond P \rightarrow \Box\Diamond P$

Here's my thinking on this problem. Goal 1: get $\Diamond\Box\Diamond P \rightarrow \Diamond P$. (Once I get that, I can add a box to each side and I'll be done.) Goal 2: get $\Diamond\Box\Diamond P \rightarrow \Diamond\Diamond\Diamond P$. (Once I get, I'll be able to get Goal 1 by using S4◊ twice.) Goal 3: get $\Box\Diamond P \rightarrow \Diamond\Diamond P$. (Once I have this, I can get goal 2 by adding a diamond to both sides.)

1.	$\Box\Diamond P \rightarrow \Diamond P$	T
2.	$\Diamond P \rightarrow \Diamond\Diamond P$	T◊
3.	$\Box\Diamond P \rightarrow \Diamond\Diamond P$	1, 2, PL (goal 3 achieved)
4.	$\Box(\Box\Diamond P \rightarrow \Diamond\Diamond P)$	3, NEC
5.	$\Box(\Box\Diamond P \rightarrow \Diamond\Diamond P) \rightarrow (\Diamond\Box\Diamond P \rightarrow \Diamond\Diamond\Diamond P)$	K◊
6.	$\Diamond\Box\Diamond P \rightarrow \Diamond\Diamond\Diamond P$	4, 5 MP (goal 2 achieved)
7.	$\Diamond\Diamond\Diamond P \rightarrow \Diamond\Diamond P$	S4◊
8.	$\Diamond\Diamond P \rightarrow \Diamond P$	S4◊
9.	$\Diamond\Box\Diamond P \rightarrow \Diamond P$	6, 7, 8, PL
10.	$\Box\Diamond\Box\Diamond P \rightarrow \Box\Diamond P$	9, NEC, K□

S5-Theorems

55. $\Box(P \vee \Diamond Q) \rightarrow (\Box P \vee \Diamond Q)$

My strategy here uses the fact from propositional logic that $\chi \rightarrow (\phi \vee \psi)$ is equivalent to $\chi \rightarrow (\sim\psi \rightarrow \phi)$. Goal 1 is to get $\Box(P \vee \Diamond Q) \rightarrow (\sim\Diamond Q \rightarrow \Box P)$. (Once I have that, I'll be able to get the formula in question by PL.) OK, how to get that? Well, $\sim\Diamond Q$ is equivalent to $\Box\sim Q$, so I'll set my Goal 2 as: $\Box(P \vee \Diamond Q) \rightarrow (\Box\sim Q \rightarrow \Box P)$. OK, how to get this? Well, this looks like a situation for the standard K strategy of agglomerating two box statements. So my first thought is to set my next goal to be: $(P \vee \Diamond Q) \rightarrow (\sim Q \rightarrow P)$. But that's not going to work. However, I *could* get $(P \vee \Diamond Q) \rightarrow (\Box\sim Q \rightarrow P)$ – that's almost a PL tautology. So that will be my goal 3. Is there a way to get from this to Goal 2? Yes – see below.

1.	$(P \vee \Diamond Q) \rightarrow (\sim\Diamond Q \rightarrow P)$	PL
2.	$\Box\sim Q \rightarrow \sim\Diamond Q$	MN

3.	$(P \vee \Diamond Q) \rightarrow (\Box \sim Q \rightarrow P)$	1, 2, PL	(Goal 3)
4.	$\Box(P \vee \Diamond Q) \rightarrow \Box(\Box \sim Q \rightarrow P)$	3, Nec, K$\Box$	
5.	$\Box(\Box \sim Q \rightarrow P) \rightarrow (\Box\Box \sim Q \rightarrow \Box P)$	K	
6.	$\Box(P \vee \Diamond Q) \rightarrow (\Box\Box \sim Q \rightarrow \Box P)$	4, 5, PL	
7.	$\Box \sim Q \rightarrow \Box\Box \sim Q$	S4	
8.	$\Box(P \vee \Diamond Q) \rightarrow (\Box \sim Q \rightarrow \Box P)$	6, 7, ,PL	(Goal 2)
9.	$\sim\Diamond Q \rightarrow \Box \sim Q$	MN	
10.	$\Box(P \vee \Diamond Q) \rightarrow (\sim\Diamond Q \rightarrow \Box P)$	8, 9, PL	
11.	$\Box(P \vee \Diamond Q) \rightarrow (\Box P \vee \Diamond Q)$	10, PL	

58. $\Box(\Box P \rightarrow \Box Q) \vee \Box(\Box Q \rightarrow \Box P)$

Plan of attack: show the PL equivalent: $\sim\Box(\Box P \rightarrow \Box Q) \rightarrow \Box(\Box Q \rightarrow \Box P)$; i.e., $\Diamond\sim(\Box P \rightarrow \Box Q) \rightarrow \Box(\Box Q \rightarrow \Box P)$; i.e., $\Diamond(\Box P \& \Diamond \sim Q) \rightarrow \Box(\Box Q \rightarrow \Box P)$:

1.	$(\Box P \& \Diamond \sim Q) \rightarrow \Box P$	PL	
2.	$\Box[(\Box P \& \Diamond \sim Q) \rightarrow \Box P]$	1, NEC	
3.	$\Diamond(\Box P \& \Diamond \sim Q) \rightarrow \Diamond\Box P$	2, K$\Diamond$	
4.	$\Diamond\Box P \rightarrow \Box P$	S5	
5.	$\Box P \rightarrow \Box\Box P$	S4	
6.	$\Diamond(\Box P \& \Diamond \sim Q) \rightarrow \Box\Box P$	3, 4, 5, PL	
7.	$\Box P \rightarrow (\Box Q \rightarrow \Box P)$	PL	
8.	$\Box\Box P \rightarrow \Box(\Box Q \rightarrow \Box P)$	7, NEC, K	
9.	$\Diamond(\Box P \& \Diamond \sim Q) \rightarrow \Box(\Box Q \rightarrow \Box P)$	6, 8, PL	
10.	$\sim(\Box P \rightarrow \sim\Diamond \sim Q) \rightarrow (\Box P \& \Diamond \sim Q)$	PL	
11.	$\Diamond\sim(\Box P \rightarrow \sim\Diamond \sim Q) \rightarrow \Diamond(\Box P \& \Diamond \sim Q)$	10, NEC, K$\Diamond$, MP	
12.	$\Diamond\sim(\Box P \rightarrow \sim\Diamond \sim Q) \rightarrow \Box(\Box Q \rightarrow \Box P)$	9, 11, PL	
13.	$\sim\Box(\Box P \rightarrow \Box Q) \rightarrow \Box(\Box Q \rightarrow \Box P)$	12, MN, dual, substitution of equivalents	
14.	$\Box(\Box P \rightarrow \Box Q) \vee \Box(\Box Q \rightarrow \Box P)$	13, PL	

59. $\Box[\Box(\Diamond P \rightarrow Q) \leftrightarrow \Box(P \rightarrow \Box Q)]$

1.	$\Box(\Diamond P \rightarrow Q) \rightarrow (\Box\Diamond P \rightarrow \Box Q)$	K	
2.	$P \rightarrow \Box\Diamond P$	B	
3.	$\Box(\Diamond P \rightarrow Q) \rightarrow (P \rightarrow \Box Q)$	1, 2, PL	
4.	$\Box\Box(\Diamond P \rightarrow Q) \rightarrow \Box(P \rightarrow \Box Q)$	3, NEC, K	
5.	$\Box(\Diamond P \rightarrow Q) \rightarrow \Box\Box(\Diamond P \rightarrow Q)$	S4	
6.	$\Box(\Diamond P \rightarrow Q) \rightarrow \Box(P \rightarrow \Box Q)$	4, 5, PL	
7.	$\Box(P \rightarrow \Box Q) \rightarrow (\Diamond P \rightarrow \Diamond\Box Q)$	K$\Diamond$	(now the other direction)

8.	$\Diamond\Box Q \rightarrow Q$	B
9.	$\Box(P \rightarrow \Box Q) \rightarrow (\Diamond P \rightarrow Q)$	7, 8, PL
10.	$\Box\Box(P \rightarrow \Box Q) \rightarrow \Box(\Diamond P \rightarrow Q)$	9, NEC, K
11.	$\Box(P \rightarrow \Box Q) \rightarrow \Box\Box(P \rightarrow \Box Q)$	S4
12.	$\Box(P \rightarrow \Box Q) \rightarrow \Box(\Diamond P \rightarrow Q)$	10, 11, PL
13.	$\Box(\Diamond P \rightarrow Q) \leftrightarrow \Box(P \rightarrow \Box Q)$	12, 6, PL
14.	$\Box[\Box(\Diamond P \rightarrow Q) \leftrightarrow \Box(P \rightarrow \Box Q)]$	13, NEC