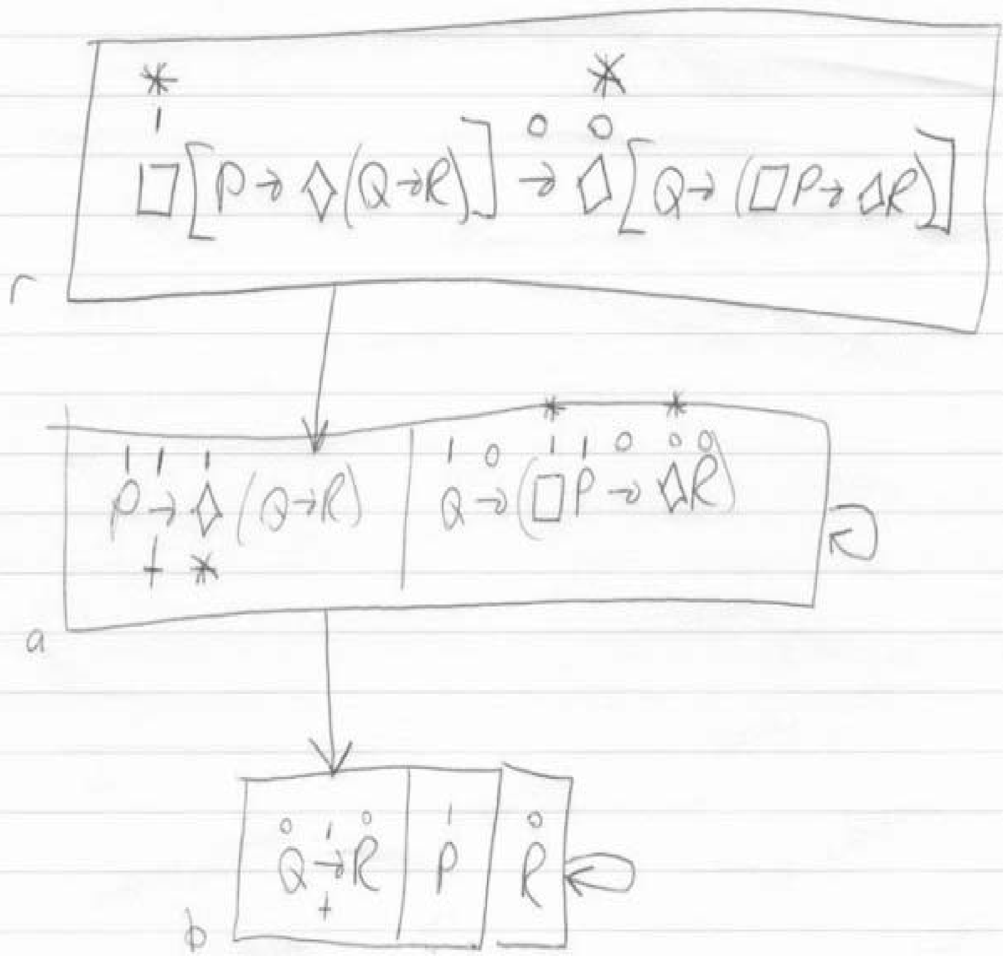
Intermediate Logic
Solutions to sample problems from the "Mixed Modal Propositional Logic Wffs" sheet

In each case, the goal is to discover in which systems the formula is valid, and in which systems it is invalid.

#1: $\Box[P\to\Diamond(Q\to R)]\to\Diamond[Q\to(\Box P\to\Diamond R)]$
Countermodel:

#1 D-Model: So, invalid in K, D



$W = \{r, a, b\}$

$R = \{\langle r,a\rangle, \langle a,b\rangle, \langle a,a\rangle, \langle b,b\rangle\}$

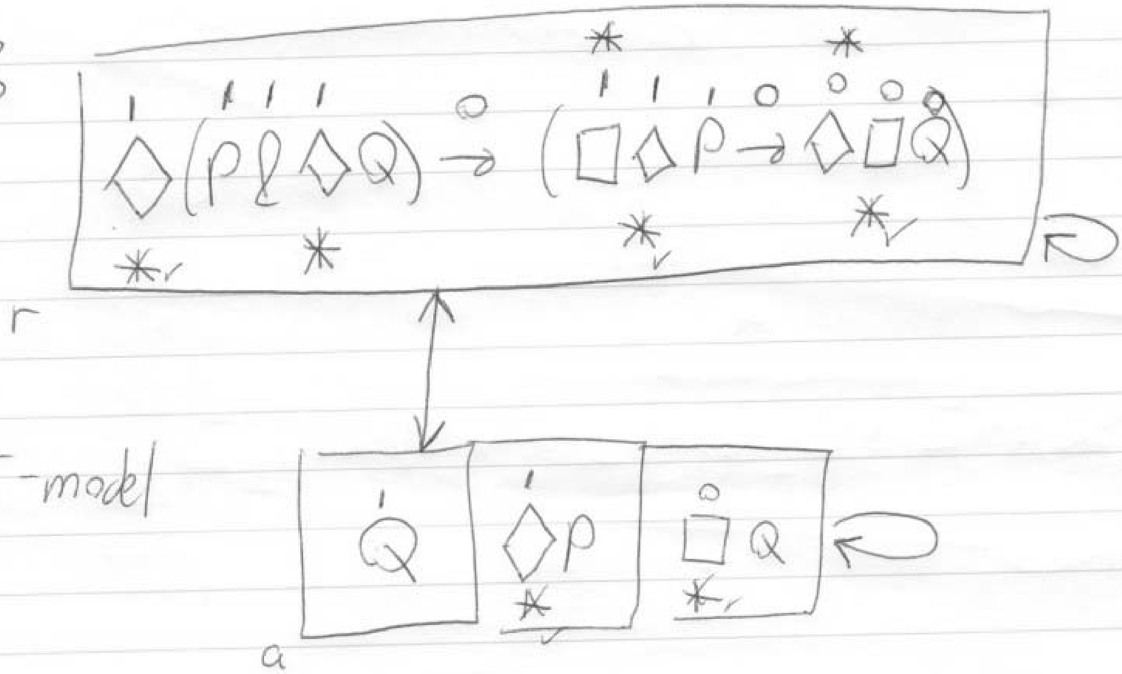
$V(P,a) = V(Q,a) = V(P,b) = 1$, all else 0

#1 continued, proof that $\Box[P\rightarrow\Diamond(Q\rightarrow R)]\rightarrow\Diamond[Q\rightarrow(\Box P\rightarrow\Diamond R)]$ is valid in T:

1. Suppose for reductio that for some T-model $\langle W,R,V\rangle$ and some $w\in W$, V(the formula,w)=0.
2. So $V(\Box[P\rightarrow\Diamond(Q\rightarrow R)],w) =1$, and...
3. ...$V(\Diamond[Q\rightarrow(\Box P\rightarrow\Diamond R)],w)=0$
4. From 3, for any v such that Rwv, $V(Q\rightarrow(\Box P\rightarrow\Diamond R),v)=0$
5. But Rww (reflexivity)
6. So $V(Q\rightarrow(\Box P\rightarrow\Diamond R),w)=0$
7. So $V(\Box P\rightarrow\Diamond R,w)=0$ (truth condition for $\rightarrow$)
8. And so, $V(\Box P,w)=1$ (truth condition for $\rightarrow$)
9. And so, since Rww, $V(P,w)=1$.
10. From 2, given Rww, we know that $V(P\rightarrow\Diamond(Q\rightarrow R),w)=1$.
11. Given 2 and the truth condition for $\rightarrow$, $V(\Diamond(Q\rightarrow R),w)=1$
12. So for some v such that Rwv, $V(Q\rightarrow R,v)=1$. Call this v "a". So we have: Rwa and ...
13. $V(Q\rightarrow R,a)=1$.
14. From 4 and 12, $V(Q\rightarrow(\Box P\rightarrow\Diamond R),a)=0$.
15. Given the truth condition for the $\rightarrow$, $V(Q,a)=1$ and...
16. $V(\Box P\rightarrow\Diamond R,a)=1$.
17. So, by the truth condition for the $\rightarrow$, $V(\Diamond R,a)=0$.
18. Thus, for each world v such that Rav, $V(R,v)=0$.
19. But Raa (reflexivity). So $V(R,a)=0$.
20. Lines 13, 15, and 19 contradict (truth condition for $\rightarrow$)

t3

S5-model



$$W = \{r, a\}$$

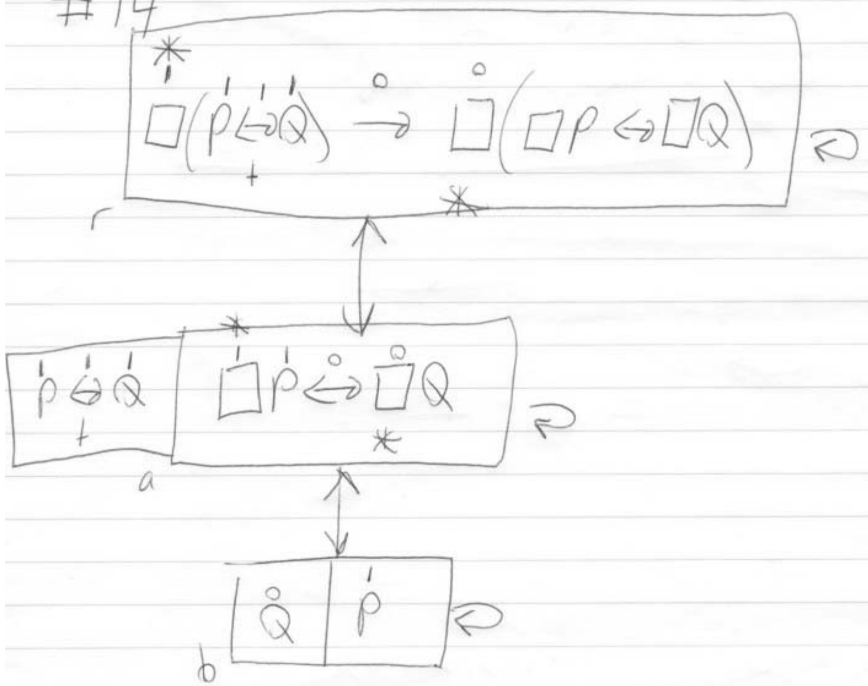
$$R = \{\langle r,r \rangle, \langle a,a \rangle, \langle r,a \rangle, \langle a,r \rangle\}$$

$$V(P,r) = V(Q,a) = 1$$

all else 0

So: invalid in all Systems

#14



This is a B-model, so the formula is invalid in B, T, D, and K

$W = \{r, a, b\}$ $R = \{\langle i,i\rangle \text{ for all } i\in W, \langle r,a\rangle, \langle a,r\rangle, \langle r,b\rangle, \langle b,r\rangle\}$

$V(P,r) = V(Q,r) = V(P,a) = V(Q,a) = V(P,b) = 1$

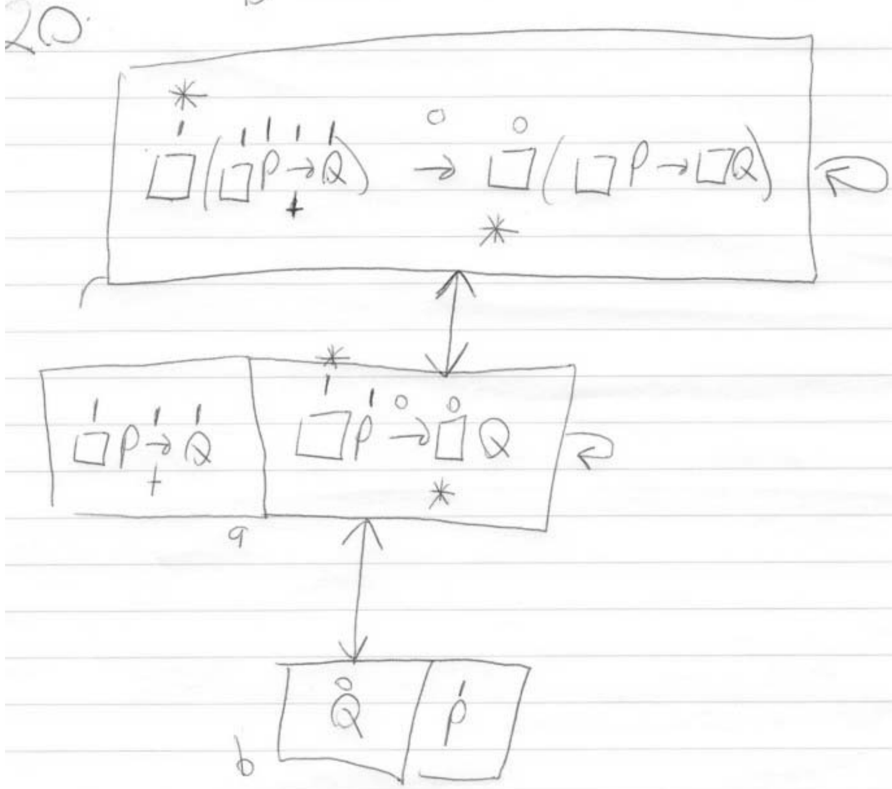
all else 0.

#14 continued, validity proof of □(P↔Q)→□(□P↔□Q) in S4:

1. Suppose for reductio that in some world w of some S4-model, ...
2. ...V(□(P↔Q),w)=1 and...
3. ...V(□(□P↔□Q),w)=0.
4. Given 3, for some world a, Rwa and V(□P↔□Q,a)=0.
5. Given 4, □P and □Q must have different truth values in world a. Without loss of generality (given the symmetry between P and Q elsewhere in the problem), let's suppose that ...
6. ...V(□P,a)=1 and ...
7. ...V(□Q,a)=0.
8. Given 7, for some world b, Rab and V(Q,b)=0.
9. Given 6, V(P,b)=1.
10. We already know that Rwa and Rab. By transitivity, Rwb.
11. But then, given 2, V(P↔Q,b)=1. This contradicts 8 and 9.

#20, $\Box(\Box P\to Q)\to\Box(\Box P\to\Box Q)$

B-model (so invalid in B, T, D, K)



$W=\{r,a,b\}$, $R=\{\langle r,r\rangle,\langle a,a\rangle,\langle b,b\rangle,\langle r,a\rangle,\langle a,r\rangle,\langle a,b\rangle,\langle b,a\rangle\}$

$V(P,r)=V(Q,r)=V(P,a)=V(Q,a)=V(P,b)=1$

all else 0

Proof that $\Box(\Box P\to Q)\to\Box(\Box P\to\Box Q)$ is valid in S4:

1. Suppose for reductio that in some world w of some S4-model, ...
2. ...$V(\Box(\Box P\to Q),w)=1$ and...
3. ...$V(\Box(\Box P\to\Box Q),w)=0$.
4. Given 3, for some world a, Rwa and $V(\Box P\to\Box Q,a)=0$.
5. By the truth condition for the $\to$, $V(\Box P,a)=1$ and...
6. ...$V(\Box Q,a)=0$.
7. Given 6, for some b, Rab and $V(Q,b)=0$.
8. Given 5, $V(P,b)=1$.
9. Since Rwa and Rab, Rwb, by transitivity.
10. But then, given 2, $V(\Box P\to Q,b)=1$.
11. Given 6, $V(\Box P,b)=0$.
12. So, for some c, Rbc and $V(P,c)=0$.
13. Since Rab and Rbc, Rac (transitivity).
14. So, given 5, $V(P,c)=1$. This contradicts 12.