Crash Course:
Paradoxes and Set Theory

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Higher-order metaphysics

4. Paradoxes

4.1 Abstract mathematics and set-theoretic foundations

Sets became central to the foundations of mathematics in the nineteenth century and early twentieth century.

Sets have members. We write them thus:

$$\{\text{Ted Sider}, \text{Barack Obama}\}$$

or

$$E = \{x | x \text{ is an even natural number}\}$$

Sets can contain other sets. There is a set with no members: $\varnothing$. Sets are unordered:

$$\{\text{Ted Sider}, \text{Barack Obama}\} = \{\text{Barack Obama}, \text{Ted Sider}\}$$

Though we can also speak of *ordered sets*:

$$\langle \text{Ted Sider}, \text{Barack Obama} \rangle \neq \langle \text{Barack Obama}, \text{Ted Sider} \rangle$$

Sets are important to foundations of mathematics because we can *construct* mathematical objects as sets.

$$
\begin{aligned}
0 \Rightarrow \varnothing \quad 1 \Rightarrow \{\varnothing\} \quad 2 \Rightarrow \{\varnothing, \{\varnothing\}\} \quad \ldots & \qquad \text{(natural numbers)} \\
\frac{1}{2} \Rightarrow \langle 1, 2 \rangle & \qquad \text{(rational numbers)} \\
\langle x, y \rangle \Rightarrow \{\{x\}, \{x, y\}\} & \qquad \text{(ordered pairs)} \\
\pi \Rightarrow \left\{ \frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \ldots \right\} & \qquad \text{(real numbers)} \\
a + bi \Rightarrow \langle a, b \rangle & \qquad \text{(complex numbers)} \\
f \Rightarrow \{\langle 0, f(0) \rangle, \langle 1, f(1) \rangle, \langle 2, f(2) \rangle, \ldots\} & \qquad \text{(functions)}
\end{aligned}
$$

4.2 Russell's paradox

But soon it was discovered that our ordinary principles for reasoning about sets are inconsistent. The following was being assumed:

Naïve comprehension For any "condition", there exists a corresponding set—a set of all and only those things that satisfy the condition

Bertrand Russell's paradox: consider the condition "is not a member of itself". Naïve comprehension implies that there exists a set R such that:

i) Anything that is *not* a member of itself *is* a member of R

ii) Anything that *is* a member of itself is *not* a member of R

The claim that R *isn't* a member of itself leads to a contradiction: if R isn't a member of itself, then by i), R would be a member of R, and so it would be a member of itself after all—contradiction.

But the claim that R *is* a member of itself also leads to a contradiction: if R is a member of itself, then by ii) it wouldn't be a member of R, and so wouldn't be a member of itself—contradiction.

More simply, Naïve comprehension implies that there exists a set, R, such that:

$$\text{for all } z, z \in R \text{ iff } z \notin z$$

But this implies, instantiating the variable z to R:

$$R \in R \text{ iff } R \notin R$$

which is a contradiction.

4.3 Axiomatic set theory and ZF

Eventually, a way of dealing with this paradox was developed which is now standard: "Zermelo-Frankel", or "ZF" set theory. Here are the main principles:

Extensionality Sets with the same members are identical

Null set There exists a set $\varnothing$ containing no members

Pairing For any sets a and b, there exists a set $\{a, b\}$ containing just a and b

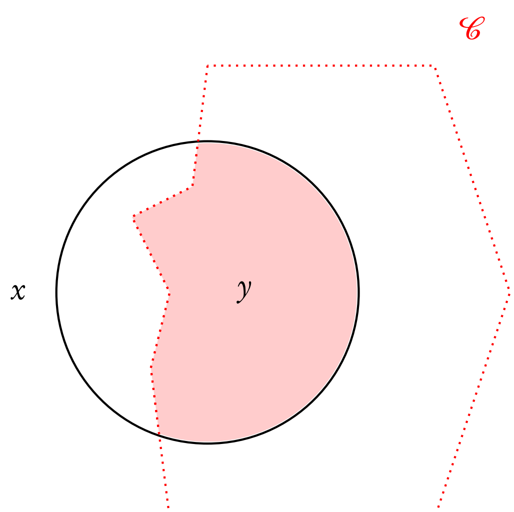
Unions For any sets a and b, there exists a set $a \cup b$ containing all and only those things that are members of a or members of b (or both)

Infinity There exists a set, A, that i) contains the null set, and i) is such that for any $a \in A$, $a \cup \{a\}$ is also a member of A. (Any such set A must be infinite, since it contains all of these sets: $\varnothing, \{\varnothing\}, \{\{\varnothing\}, \varnothing\}, \ldots.$)

Power set For any set, A, there exists a set containing all and only the subsets of A (this is called A's "power set")

Separation Suppose some set x exists, and let $\mathscr{C}$ be any condition. Then there exists a set y consisting of all and only the members of x that satisfy $\mathscr{C}$.

Picture of separation:



• Separation doesn't lead to Russell's paradox. If you start with a set, x, you can choose the condition "is not a member of itself", and conclude by Separation that there exists a *subset* of x, call it y, that contains all and only the non-self-membered members of x:

$$\text{For all } z : z \in y \text{ iff } z \in x \text{ and } z \notin z$$

But this doesn't lead to a contradiction. It does imply this:

$$y \in y \text{ iff } y \in x \text{ and } y \notin y$$

But now we can consistently suppose that $y \notin y$, so long as $y \notin x$.

• Does there exist a *universal set*, U, containing *every* set? If so then by separation:

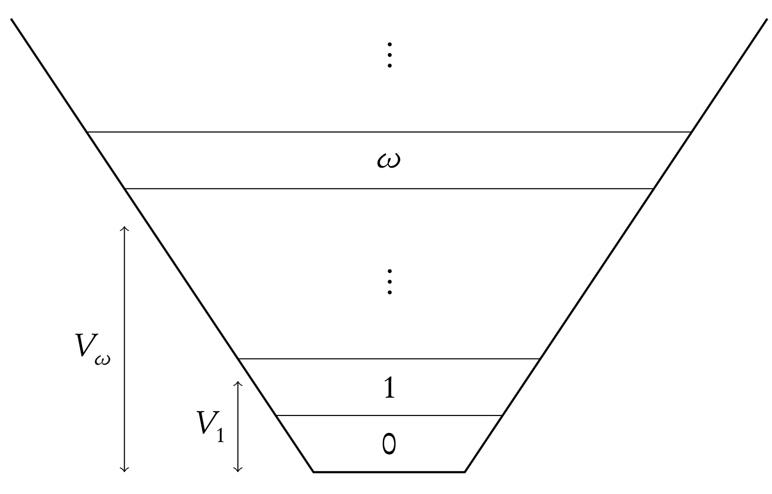
$$\text{For all } z : z \in R \text{ iff } z \in U \text{ and } z \notin z$$

But since *every* z is a member of U, this implies:

$$\text{For all } z : z \in R \text{ iff } z \notin z$$

which is Russell's contradiction. So no, there is no universal set in ZF.

Picture of the sets according to ZF—the "iterative hierarchy":



V_0 is the set of urelements

My statement of separation was nonrigorous; and a rigorous statement differs depending on whether we are using first- or second-order logic:

$$\forall x \exists y \forall z \big(z \in y \leftrightarrow (z \in x \wedge A)\big) \quad \text{(first-order separation \textit{schema})}$$

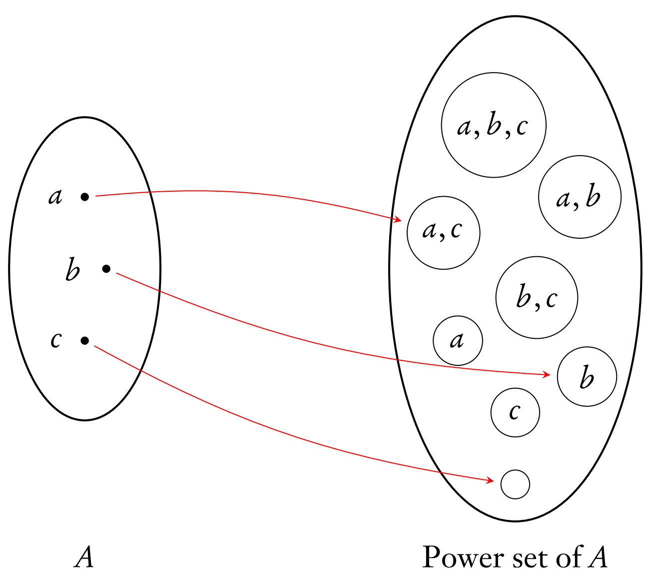
$$\forall x \forall X \exists y \forall z \big(z \in y \leftrightarrow (z \in x \wedge Xz)\big)$$
$$\text{(second-order separation \textit{principle})}$$

As with induction, the second-order version is stronger, and would seem to be what we really want.

4.4 Other paradoxes, other solutions

Russell-type paradoxes arise for entities other than sets: the property of *being a non-self-exemplifying property*, the predicate ‘heterological’, meaning “does not apply to itself”.

To lead up to another paradox, consider Cantor’s proof that the power set of a set is always *larger* than that set, meaning that if you take each member of the set and pair it with a unique member of the powerset, some members of the powerset will be left over, not paired with any member of the set:



Cantor's argument:

> Let f be any function that maps every member of a set, A, to some subset of A. Form the set, D, of members of A that are mapped to sets that don't contain them:
>
> $$\text{for any } x,\ x \in D \text{ iff: } x \in A \text{ and } x \notin f(x) \qquad (*)$$
>
> Now, suppose for reductio that for every subset, X, of A, some member of A is mapped to X by f. Then some member, d, of A is mapped to D by f; i.e., $f(d) = D$.
>
> But there can be no such d. By (*),
>
> $$d \in D \text{ iff: } d \in A \text{ and } d \notin f(d)$$
>
> But $f(d) = D$, so:
>
> $$d \in D \text{ iff: } d \in A \text{ and } d \notin D$$
>
> But $d \in A$, so:
>
> $$d \in D \text{ iff } d \notin D$$
>
> which is a contradiction.

Back, now, to propositions. It would be natural to assume that for any set, S, of propositions, there exists a unique proposition that *some god is currently entertaining all and only the propositions that are members of* S. But that can't be: it would mean that there are at least as many propositions as there are sets of propositions.

Back to set theory and Russell's paradox. We discussed the ZF solution, but there are others:

- Deny classical logic; $R \in R \leftrightarrow R \notin R$ isn't a contradiction.
- Accept a different set theory, e.g., Quine's "New foundations".
- Indefinite extensibility: when you introduce the Russell set, you expand the domain of quantification; there is no such thing as absolutely unrestricted quantification.
- "Syntactic" solutions. (Russell's own solution was like this.) Rough idea: come up with a syntax in which attempts to construct the paradox aren't grammatical. E.g., in second-order logic you can't grammatically attribute a property to itself: FF isn't a formula.