Logicism: Frege

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Philosophy of Mathematics

Basic problem: mathematics seems to be both a priori and synthetic, which is puzzling. Kant tried to make the combination work; Mill tried denying a prioricity. Logicists try the final option: mathematics is analytic.

1. Frege

A German mathematician and philosopher from the late nineteenth and early twentieth century. He was mostly unknown during his day, but is now regarded as a major figure in philosophy and logic. He is the main inventor of modern predicate logic.

2. Mathematics is analytic

Frege's conception of analyticity:

> A proposition is analytic if either it is a 'general logical law or definition' or it has a proof that relies only on such general logical laws and definitions. (Shapiro p. 109)

3. Logic for Frege

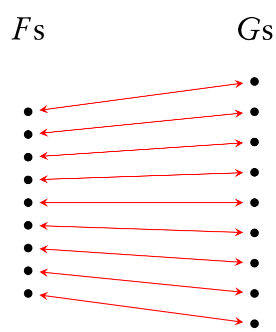
Frege's logic was "second order". In addition to sentences like $\exists xFx$ and $\forall x\exists yRxy$, he also accepted sentences like $\exists FFa$ and $\forall R(\exists x\forall yRxy \rightarrow \forall y\exists xRxy)$, in which there are quantifiers for "concepts".

4. The-number-of, and Hume's Principle

Where F is a one-place predicate, Frege uses the phrase "the number of Fs", which understood to refer to an entity. He accepts this principle governing the phrase:

Hume's Principle the number of the Fs = the number of the Gs if and only if F and G are equinumerous.

The Fs and the Gs are equinumerous if each F can be paired with a unique G and vice versa:



That is, there is a one-to-one correspondence between the Fs and the Gs:

The Fs and the Gs are equinumerous iff for some R, each F bears R to exactly one G, and for each G there is exactly one F that bears R to it; i.e.:

$$\exists R(\forall x(Fx \rightarrow \exists y(Gy \wedge Rxy \wedge \forall z((Gy \wedge Rxz) \rightarrow y = z)))\wedge$$
$$\forall x(Gx \rightarrow \exists y(Fy \wedge Ryx \wedge \forall z((Gy \wedge Rzx) \rightarrow y = z))))$$

5. Definition of individual numbers

0 is defined as the number of things that are not self-identical

1 is defined as the number of things that are identical to 0

2 is defined as the number of things that are either identical to 0 or identical to 1

6. Definition of successor

> n is a successor of m if and only if for some F and some x: i) Fx, ii) $n =$ the number of Fs, and iii) $m =$ the number of Fs-that-are-not-identical-to-x

This implies, for example, that 1 is a successor of `0`:

i) `0` = the number of things that are not self-identical (def)

ii) 1 = the number of things that are identical to `0` (def)

iii) `0` = `0` (logic)

iv) The concept thing-that-is-not-self-identical is equinumerous with the concept thing-that-is-identical-to-`0`-and-not-identical-to-`0` (logic)

v) The number of things that are not self-identical = the number of things that are identical to `0` and not identical to `0` (iv, Hume's Principle)

vi) `0` = the number of things that are identical to `0` and not identical to `0` (i, v, logic)

vii) `0` = `0`, and 1 = the number of things that are identical to `0`, and `0` = the number of things that are identical to `0` and not identical to `0` (iii, ii, vi, logic)

viii) For some F and some x, Fx, 1 = the number of Fs, and `0` = the number of Fs that are not identical to x (vii, logic)

> (F is: thing-that-is-identical-to-`0`, and x is: `0`)

ix) 1 is a successor of `0` (viii, def)

7. Definition of natural number

This definition is no good, because it's not rigorous:

> The numbers are 0, 1, 2, and so on

This is no good because sometimes there are infinitely many Fs:

> n is a natural number iff for some F, n is the number of Fs

This doesn't work because it uses an undefined notion of 'finite':

> n is a natural number iff for some F, there are *finitely many* Fs and n is the number of Fs

Here is Frege's definition:

> n is a natural number if and only if: for any F, if 0 has F, and if whenever some m has F, so does every successor of m, then n is F.
>
> In other words: a natural number is anything that has every concept that holds of zero and is closed under successor.
>
> In yet other words: the class of numbers is the smallest class that obeys the principle of induction.

8. Caesar problem

If arithmetic is to be derived from pure logic, all principles in the derivation need to be definitions or laws of logic. Hume's Principle appears to be neither.

Or might it be regarded as a definition of the idea of 'the number of'? No, thought Frege, since it doesn't tell us the truth values of all sentences of the form "the number of Fs $= t$"; it doesn't tell us whether "the number of Fs $=$ Julius Caesar", for example.

9. Extensions

So Frege tried to give a genuine definition of 'the number of', from which Hume's Principle could be derived:

> The number of Fs is defined as the extension of the concept: *being a concept*, G, *such that* G *is equinumerous to* F

This appeals to *extensions of concepts*, which Frege thought of as being a matter of logic. Extensions obey, according to Frege, this principle:

Frege's Basic Law V The extension of F = the extension of G if and only if: for any object x, x has F if and only if x has G

He then used this law to derive Hume's Principle:

> The number of Fs = the number of Gs …
>
> …iff the extension of *being equinumerous to* F = the extension of *being equinumerous to* G (def of number of)
>
> …iff for any concept, H, F is equinumerous to H iff G is equinumerous to H (Basic Law V)
>
> …iff F and G are equinumerous (equinumerosity is an equivalence relation)

10. Reduction of arithmetic

Frege then proved that the (second-order) Peano axioms are derivable from his definitions in his (second-order) logic, which included Basic Law V.

11. Russell's objection

Russell showed that Basic Law V is inconsistent! Let:

> R = the concept of *being the extension of some concept you don't possess*

That is:

(*) x has R iff for some concept F, $x =$ the extension of F and x does not have F

We can now argue as follows:

1. Let $r =$ the extension of R. Question: does r have R?
2. Suppose *yes*: r has R. This leads to a contradiction:
   (a) Then for some F, $r =$ the extension of F and <u>r does not have F</u> (2, *)
   (b) For any y, y has R iff y has F (1, 2a, Basic Law V)
   (c) <u>r has F</u> (2, 2b)
3. Suppose *no*: <u>r does not have R</u>. This too leads to a contradiction:
   (a) $r =$ the extension of R and r does not have R (1, 3)
   (b) For some concept F (namely: R), $r =$ the extension of F and r does not have F (3a)
   (c) <u>r has R</u> (3b, *)

Either way we get a contradiction. Thus the very foundation of Frege's system is demonstrably false. Shapiro writes:

> Frege took this paradox to be devastating to his logicist programme. Nevertheless, he sent Russell a gracious reply, almost immediately:
>
> > Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic… [The matter is] all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish… In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance. (van Heijenoort 1967: 127–8)
>
> In the same letter, Frege gave a more accurate formulation of the paradox. After some attempts to recover from the blow, Frege abandoned his logicist project, left in ruins.