Example validity and invalidity proofs. (solutions next page)

1. $\vDash \forall x(Fx \vee {\sim}Fx)$
2. $\vDash Laf(a) \to \exists x Lxf(x)$
3. $\vDash \exists x[Fx \& \forall y(Fy \to x=y) \& Gx] \leftrightarrow G\iota xFx$

4. $\nvDash \forall x(Fx \to Fg(x))$
5. $\{\forall x(Fx \to Fg(x)), \forall x(Fx \to Gx)\} \nvDash \forall x(Gx \to Gg(x))$

Solutions: example validity and invalidity proofs. NOTE: I'll do these relatively quickly, and with little comment. For a better guide to writing out your own proofs, check the solutions to the official homework problems. Use these solutions just to check whether you yourself have got the right solution.

1. $\vDash \forall x(Fx \vee {\sim}Fx)$

Take any model; take any assignment g based on that model, and suppose for reductio that $V_g(\forall x(Fx \vee {\sim}Fx))=0$, i.e., that for some $u \in D$, $V_{g(u/x)}(Fx \vee {\sim}Fx)=0$. Call this u "u". since this disjunction is false, each disjunct must be false -- $V_{g(u/x)}(Fx)=0$ and $V_{g(u/x)}({\sim}Fx)=0$. Given the second thing, $V_{g(u/x)}(Fx)=1$; but that's impossible.

2. $\vDash Laf(a) \rightarrow \exists xLxf(x)$

Suppose for reductio that in some model and for some g, $V_g(Laf(a))=1$ and $V_g(\exists xLxf(x))=0$. Given the first fact, $\langle [a]_g,[f(a)]_g \rangle \in I(L)$, and so $\langle I(a),I(f)(I(a)) \rangle \in I(L)$. Given the second fact, for each $u \in D$, $V_{g(u/x)}(Lxf(x))=0$. Letting u be I(a), we have: $V_{g(I(a)/x)}(Lxf(x))=0$. Thus, $\langle [x]_{g(I(a)/x)},[f(x)]_{g(I(a)/x)} \rangle \notin I(L)$. But $[x]_{g(I(a)/x)}$ is just I(a); and $[f(x)]_{g(I(a)/x)}$ is $I(f)([x]_{g(I(a)/x)})$ – i.e., I(f)(I(a)). Thus, we have $\langle I(a),I(f)(I(a)) \rangle \notin I(L)$. contradiction.

3. $\vDash \exists x[Fx \& \forall y(Fy \rightarrow x=y) \& Gx] \leftrightarrow G\iota xFx$

We must show that in any model, and for any g in that model, V_g(the whole thing)=1 – i.e., that $V_g(\exists x[Fx \& \forall y(Fy \rightarrow x=y) \& Gx])$ is the same as $V_g(G\iota xFx)$.

Either I(F) has exactly one member, or it doesn't. We'll show that in each case, these two formulas have the same truth value. Let's take the second case first.

If I(F) does not have exactly one member (i.e., it has no members, or two or more members), then $[\iota xFx]_g$ is d_0; since d_0 is not in the extension of any predicates, $V_g(G\iota xFx)=0$. But in this case, $V_g(\exists x[Fx \& \forall y(Fy \rightarrow x=y) \& Gx])$ is also 0. For suppose otherwise – suppose $V_g(\exists x[Fx \& \forall y(Fy \rightarrow x=y) \& Gx])=1$. Then there is some $u \in D$ such that $V_{g(u/x)}(Fx \& \forall y(Fy \rightarrow x=y) \& Gx)=1$. Call this u "u". So, $V_{g(u/x)}(Fx)=1$ and $\mathbf{V_{g(u/x)}(\forall y(Fy \rightarrow x=y))=1}$. Given the first, $u \in I(F)$. So we know that there is at least one thing in I(F) – u. And so, since we're here assuming that I(F) does not have exactly one member, it follows that there must be some *other* member of I(F) – call it o. Given the second (the thing in boldface), for any $v \in D$, $V_{g(u/x)(v/y)}(Fy \rightarrow x=y)=1$. Letting v be 0, we have $V_{g(u/x)(o/y)}(Fy \rightarrow x=y)=1$. But $V_{g(u/x)(o/y)}(Fy)=1$ since $[y]_{g(u/x)(o/y)}$=o and o is in I(F). So $V_{g(u/x)(o/y)}(x=y)=1$, and so $[x]_{g(u/x)(o/y)}=[y]_{g(u/x)(o/y)}$, and so u=o. But o is some object *other* than u. contradiction.

Now for the first case: assume now that I(F) has exactly one member. Call that member u. In that case, $[\iota xFx]_g$ will be u, and so $V_g(G\iota xFx)=1$ iff $u\in I(G)$. So we need only show that $V_g(\exists x[Fx\&\forall y(Fy\rightarrow x=y)\&Gx])=1$ iff this object $u\in I(G)$. Well, $V_g(\exists x[Fx\&\forall y(Fy\rightarrow x=y)\&Gx])=1$ iff for some $v\in D$, $V_{g(v/x)}(\exists x[Fx\&\forall y(Fy\rightarrow x=y)\&Gx])=1$. And this is true iff $u\in I(G)$; because if $u\in I(G)$ then we can set v to u, and this long formula will come out true; and if $u\notin I(G)$ then there's nothing else we can set v to be. (I'm leaving some reasoning out here, but it's like what's in the previous paragraph. Basically, since there's exactly one member of F's extension, namely u, then the first two conjuncts, Fx and $\forall x(Fy\rightarrow x=y)$ come out true iff we assign u to x.)

4. $\nvDash\forall x(Fx\rightarrow Fg(x))$

D={0,1)
I(F)={0}
I(g) = the function that assigns 0 to 1 and assigns 1 to 1.

5. $\{\forall x(Fx\rightarrow Fg(x)),\forall x(Fx\rightarrow Gx)\}\nvDash\forall x(Gx\rightarrow Gg(x))$

D = {0,1,2}
I(F)={0}
I(G)={0,1}
I(g)= the function that assigns 0 to 0 and assigns 1 to 2 and assigns 2 to 2