Math Logic
Homework #7a (Chapter 14)

1. A theory is *complete* iff for every sentence, S (in its language), either S or ~S is a theorem of the theory. Prove that Q is not complete. (Hint: use the hint to exercise 14.2 on p. 169; that will show you how to prove that certain sentences are not theorems of Q.)

2. Prove

(*) For any term, t, in the language of arithmetic, there is a unique number, i, such that $\vdash_Q t=\mathbf{i}$

Hint: use strong induction on the number of function symbols occurring in t.

3. Prove the following:

Let φ be any quantifier-free sentence in the language of arithmetic that is true in the standard model. Show that φ is a theorem of Q.

The lemmas proved in chapter 14 should be helpful, as well as (*) from problem # 1.

Hint: it will be easier if you prove the following assertion (by strong induction on the number of connectives):

Where φ is any quantifier-free sentence in the language of arithmetic containing only the boolean connectives ~ and &,

i) if φ is true in the standard model then $\vdash_Q \varphi$, AND
ii) if φ is not true in the standard model then $\vdash_Q \sim\varphi$

The desired result follows from this, since every boolean connective can be defined in terms of ~ and &.