Boolos

Ted Sider
Higher-order metaphysics

1. Foil: Quine

Higher-order logic is "set theory in sheep's clothing" (Quine, 1970, pp. 66–8).

That is: '$\exists X Xa$' means, or anyway is true if and only if, $\exists x(x$ is a set and $a \in x)$, or $\exists x(x$ is a property and a instantiates $x)$, or something like that.

(Presupposition of the debate: debates about what $\exists$ are in good standing.)

Quine's view doesn't follow merely from the fact that the standard semantics for first-order logic is set-theoretic.

(Regarding Quine's arguments, see, for example, Boolos (1975); Turner (2015))

2. Boolos

Boolos rejects Quine's view; defends "ontologically innocent" (monadic) second-order quantification.

2.1 Nonfirstorderizable sentences

Second-order quantifiers needed to "symbolize" certain sentences of natural language.

(GK) Some critics admire only one another

This sentence (the "Geach-Kaplan" sentence) can't be symbolized in first-order logic (i.e., there is no sentence of first-order logic that has the "right" truth value in all interpretations), but it can be symbolized in second-order logic:

$$\exists X(\exists x X x \wedge \forall x \forall y((X x \wedge A x y) \to (x \neq y \wedge X y))$$

New examples from Boolos:

(G) There are some horses that are all faster than Zev and also faster than the sire of any horse that is slower than all of them.

(I) There are some gunslingers each of whom has shot the right foot of at least one of the others.

Also, sentences that *are* first-orderizable, but whose most "natural" symbolizations are second-order:

(Q) There are some monuments in Italy of which no one tourist has seen all.

More natural symbolization:

$$\exists X(\exists x Xx \wedge \forall x(Xx \rightarrow Mx) \wedge \sim\exists x(Tx \wedge \forall y(Xy \rightarrow Sxy)))$$

Equivalent though less natural symbolization:

$$\exists x Mx \wedge \sim x(Tx \wedge \forall y(My \rightarrow Sxy))$$

("No tourist has seen all the monuments in Italy").

2.2 The plurals interpretation

- "There are some things such that …" $\Rightarrow \exists X$
- "There are some critics such that …" $\Rightarrow \exists X(\forall x(Xx \rightarrow Cx) \wedge \dots$
- Certain uses of 'they', 'them', etc., correspond to recurrence of second-order variables
- "is one of" corresponds to second-order predication: Xy corresponds to "y is one of X

> … neither the use of plurals nor the employment of second-order logic commits us to the existence of extra items beyond those to which we are already committed. We need not construe second-order quantifiers as ranging over any-thing other than the objects over which our first-order quantifiers range, and, in the absence of other reasons for thinking so, we need not think that there are collections of (say) Cheerios, in addition to the Cheerios. Ontological commitment is carried by our *first*-order quantifiers; a second-order quantifier needn't be taken to be a kind of first-order quantifier in disguise, having items of a special kind, collections, in its range. It is not as though there were two sorts of things in the world, individuals, and collections of them, which our first- and second-order variables, respectively, range over and which our singular and plural forms, respectively, denote. There are, rather, two (at least) different ways of referring to the same things, among which there may well be many, many collections. (p. 449)

2.3 Quantifying over all sets

Another argument Boolos gives emerges from the following problem with using second-order quantifiers to make statements about sets:

1. Suppose you can use second-order logic, obeying its usual rules, in combination with first-order quantifiers that range unrestrictedly over all sets.
2. And suppose further that $\exists X$ means "for some set…"
3. Since part of the usual logic is the principle of comprehension, the following is true:
$$\exists X \forall y(Xy \leftrightarrow y \notin y)$$
4. But if $\forall y$ ranges over all sets, and Xy means that x is a member of set Y, then this sentence leads to Russell's paradox

Boolos (1975) used to accept 2 and deny 1. But here he says that we *need* to use second-order logic while quantifying over all sets, in order to formulate a decent ZF set theory. In first-order ZF, we need an axiom schema of separation.

> It is, I think, clear that our decision to rest content with a set theory formulated in the first-order predicate calculus with identity… must be regarded as a compromise, as falling short of saying all that we might hope to say. Whatever our reasons for adopting Zermelo-Fraenkel set theory in its usual formulation may be, we accept this theory because we accept a stronger theory consisting of a *finite* number of principles, among them some for whose complete expression second-order formulas are required. We ought to be able to formulate a theory that reflects our beliefs. (p. 441)

We can do this with a second-order set theory, and a single principle of separation.

So according to Boolos, even when quantifying over all sets we accept:

$$\exists X \forall y(Xy \leftrightarrow y \notin y)$$

This doesn't lead to a paradox. The first-order analog

$$\exists x \forall y(y \in x \leftrightarrow y \notin y)$$

is paradoxical since you can let y be x, resulting in:

$$x \in x \leftrightarrow x \notin x$$

But in the second order sentence you can't let y be X; the result isn't grammatical:

$$XX \leftrightarrow X \notin X$$

- Argument can't be that first-order ZF is inadequate because there are, intuitively, truths in that language that we can't state.
- Regarding the inability to state "what we believe": we might believe things that don't correspond to anything in the world, or are even incoherent.
- Better argument: second-order ZF is a *better theory*, which gives us reason to accept any conceptual resources needed to state it.
- Compare argument that we should posit mass and charge in order to state laws of dynamics.
- What about the rejoinder that even for second-order logic we still need schemas (or a rule of substitution) in the logic, so we still can't state good laws?
- Reply: there, any purportedly better theory (e.g., bringing in proper classes) will have the very same purported defect. Compare the question of "law-laws".
- First-orderist reply: the schemas, as a group, are good laws because they're syntactically parallel.

2.4 Argument from formal semantics

> It may be suggested that [sentences with plural quantifiers] like (i) are intelligible, but only because we antecedently understand statements about collections, totalities, or sets, and that these sentences are to be analyzed as claims about the existence of certain collections, etc. Thus "There are some gunslingers ..." is to be analyzed as the claim that there is a collection of gunslingers .... The suggestion may arise from the thought that any precise and adequate semantics for natural language must be interpretable in set theory (with individuals). How else, one may wonder, is one to give an account of the semantics of plurals? (p. 446)

Against the last last thought in the quotation:

> One should not confuse the question whether certain sentences of our language containing plurals are intelligible with the question whether one can give a semantic theory for those sentences. In view of the work of Tarski, it should not automatically be expected that we can give an adequate semantics for English—whatever that might be—in English. Nothing whatever about the intelligibility of those sentences would follow from the fact that a systematic semantics for them cannot be given in set theory. After all, the semantics of the language of ZF itself cannot be given in ZF. (p. 446)

2.5 Intuitions about synonymy and implication

Further argument against the set-theoretic view

> the claim conflicts with a strong intuition, which I for one am loath to abandon, about the meaning of English sentences of the form "There are some *A*s of which every *B* is one," viz. that any sentence of this form means the same thing as the corresponding sentence of the form "There are some *A*s and every *B* is an *A*." If so, [the sentence "There are some sets of which every set that is not a member of itself is one"] is simply synonymous with the trivial truth "There are some sets and every set that is not a member of itself is a set," and therefore does not entail the existence of an overly large set. (p. 447)

He wouldn't need to claim that 'There are some *A*s of which every *B* is one' and 'There are some *A*s and every *B* is an *A*' are synonymous; he could instead say that it's a matter of meaning that the second implies the first.

Similar argument:

> It is haywire to think that when you have some Cheerios, you are eating a *set*-what you're doing is: eating The Cheerios. Maybe there are some reasons for thinking there is such a set-there are, after all, $> 10^{60}$ ways to divide the Cheerios into two portions—but it doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all. (pp. 448–9)

2.6 Semantics versus metaphysics

References

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