Theorem 1 If Γ is any set of modal wffs and $\mathscr{M}$ is an MPL-model in which each wff in Γ is valid, then every theorem of $K+\Gamma$ is valid in $\mathscr{M}$.

Lemma 2 All instances of the PL- and K-axiom schemas are valid in all MPL-models

Lemma 3 For every MPL-model, $\mathscr{M}$, MP and NEC preserve validity in $\mathscr{M}$

Crucial feature canonical models will be shown to have:

If a formula is valid in the canonical model for S, then it is a theorem of S

New definition of S-provability-from: A wff ϕ is provable in system S from a set Γ ("$\Gamma \vdash_S \phi$") iff for some $\gamma_1 \dots \gamma_n \in \Gamma$, $\vdash_S (\gamma_1 \wedge \dots \wedge \gamma_n) \to \phi$ (or else $\Gamma = \varnothing$ and $\vdash_S \phi$)

Definition of S-consistency: A set of wffs Γ is S-inconsistent iff $\Gamma \vdash_S \bot$. Γ is S-consistent iff it is not S-inconsistent.

Definition of canonical model: The canonical model for system S is the MPL-model $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ where:

- $\mathscr{W}$ is the set of all maximal S-consistent sets of wffs
- $\mathscr{R}ww'$ iff $\Box^-(w) \subseteq w'$
- $\mathscr{I}(\alpha, w) = 1$ iff $\alpha \in w$, for each sentence letter α and each $w \in \mathscr{W}$
- $\Box^-(\Delta)$ is defined as the set of wffs ϕ such that $\Box\phi$ is a member of Δ

Theorem 4 If Δ is an S-consistent set of MPL-wffs, then there exists some maximal S-consistent set of MPL-wffs, Γ, such that $\Delta \subseteq \Gamma$.

Lemma 5 Where Γ is any maximal S-consistent set of MPL-wffs:

5a for any MPL-wff ϕ, exactly one of ϕ, $\sim\phi$ is a member of Γ

5b $\phi \to \psi \in \Gamma$ iff either $\phi \notin \Gamma$ or $\psi \in \Gamma$

- If $\Gamma \vdash_{PL} \phi$, then $\gamma_1 \dots \gamma_n \vdash_{PL} \phi$, for some $\gamma_1 \dots \gamma_n \in \Gamma$ (or else $\vdash_{PL} \phi$) (lemma **??**)
- "Excluded middle MP": $\phi \to \psi, \sim\phi \to \psi \vdash_{PL} \psi$
- "Ex falso quodlibet": $\phi, \sim\phi \vdash_{PL} \psi$
- Modus ponens: $\phi, \phi \to \psi \vdash_{PL} \psi$
- "Negated conditional": $\sim(\phi \to \psi) \vdash_{PL} \phi$ and $\sim(\phi \to \psi) \vdash_{PL} \sim\psi$
- If $\phi \in \Gamma$, then $\Gamma \vdash_{PL} \phi$

- Cut for PL
- The deduction theorem for PL

Deduction theorem for MPL: For each of our modal systems S (and given our new definition of provability from a set), if $\Gamma \cup \{\phi\} \vdash_S \psi$, then $\Gamma \vdash_S \phi \to \psi$.

5c if $\vdash_S \phi$, then $\phi \in \Gamma$

5d if $\vdash_S \phi \to \psi$ and $\phi \in \Gamma$, then $\psi \in \Gamma$

Proof. For 5c, if $\vdash_S \phi$, then $\vdash_S (\sim\phi \to \bot)$, since S includes PL. Since Γ is S-consistent, $\sim\phi \notin \Gamma$; and so, since Γ is maximal, $\phi \in \Gamma$. For 5d, use lemmas 5c and 5b. ■

Lemma 6 If Δ is a maximal S-consistent set of wffs containing $\sim\Box\phi$, then there exists a maximal S-consistent set of wffs Γ such that $\Box^-(\Delta) \subseteq \Gamma$ and $\sim\phi \in \Gamma$

Theorem 7 Where $\mathscr{M}$ $(= \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle)$ is the canonical model for any normal modal system, S, for any wff ϕ and any $w \in \mathscr{W}$, $V_{\mathscr{M}}(\phi, w) = 1$ iff $\phi \in w$.

Corollary 8 ϕ is valid in the canonical model for S iff $\vdash_S \phi$.