

Some notes on proofs and variables

1.

Let me introduce two approaches to proofs, call them Approach A and Approach B. People often use Approach A, but Approach B is better. I'll illustrate with the proof from section 2.4 that $\wedge I$ preserves sequent validity.

In that proof we are showing that if two sequents $\Gamma \Rightarrow \phi$ and $\Delta \Rightarrow \psi$ are valid (these are the inputs to $\wedge I$), then the sequent $\Gamma \cup \Delta \Rightarrow \phi \wedge \psi$ (the output of the rule) must also be valid. ($\Gamma \cup \Delta$ is the set containing all the members of Γ and also all the members of Δ .) So we have two main premises to work with:

P₁. $\Gamma \Rightarrow \phi$ is a valid sequent

P₂. $\Delta \Rightarrow \psi$ is a valid sequent

Plus we have the definitions of 'valid sequent' and 'semantic implication' ($\models$) to work with. What we must do is use these premises and definitions to reach the conclusion that $\Gamma \cup \Delta \Rightarrow \phi \wedge \psi$ is a valid sequent.

Approach A to this problem goes like this:

Premises P₁ and P₂ tell us, given the definition of a valid sequent, that $\Gamma \models \phi$ and that $\Delta \models \psi$. And so, given the definition of semantic implication ($\models$), we know that:

1. For every interpretation in which every member of Γ is true, ϕ is true
2. For every interpretation in which every member of Δ is true, ψ is true

It therefore follows that:

3. For every interpretation in which every member of $\Gamma \cup \Delta$ is true, both ϕ and ψ are true.

But the truth-condition for $\wedge$ tells us that:

4. For every interpretation in which both ϕ and ψ are true, their conjunction $\phi \wedge \psi$ is also true.

And from 3 and 4 it follows that:

5. For every interpretation in which every member of $\Gamma \cup \Delta$ is true, $\phi \wedge \psi$ is true.

Thus $\Gamma \cup \Delta \models \phi \wedge \psi$, and so, $\Gamma \cup \Delta \Rightarrow \phi \wedge \psi$ is a valid sequent. We're done.

Approach B (the approach I use in the book) goes like this:

What we need to show is that $\Gamma, \Delta \Rightarrow \phi \wedge \chi$ is a valid sequent. That is, we must show that $\Gamma \cup \Delta \models \phi \wedge \chi$. That is, we must show that:

Goal For every interpretation in which every member of $\Gamma \cup \Delta$ is true, $\phi \wedge \psi$ is true.

(This is the same statement as claim 5 that is made at the end of the Approach A proof, but here it isn't a *claim* since we haven't yet established it; rather, we're announcing that it's our goal to prove it.) So in order to establish Goal, we begin by letting I be an arbitrary interpretation, and we suppose that:

- i. every member of $\Gamma \cup \Delta$ is true in I

And now we must try to show that i implies that $\phi \wedge \psi$ is also true in I . Here we go:

Given the definition of set-union ("U"), we can conclude from i that:

- ii. Every member of Γ is true in I

But P1, together with the definition of 'valid sequent', tells us that $\Gamma \models \phi$, and thus, given ii and the definition of 'semantic implication', we know that:

- iii. ϕ is true in I

(To say that $\Gamma \models \phi$ is to say that in every interpretation in which every member of Γ is true, ϕ is also true.) Similarly, i implies:

iv. Every member of Δ is true in I

But P_2 tells us that $\Delta \models \phi$, which, given iv, tells us that:

v. ϕ is true in I

But given iii and v, and the truth-condition for the $\wedge$:

vi. $\phi \wedge \psi$ is true in I

So: we've shown that the assumption of i (where I was arbitrarily chosen) leads to vi. Thus we've established our Goal. We're done.

The difference between the two approaches is as follows.

In approach A, we began by unpacking what premises P_1 and P_2 told us, which gave us claims 1 and 2. Then we made a series of inferences: 1 and 2 together imply 3; but the t.c. for $\wedge$ tells us 4; and 3 and 4 together imply 5. But notice that each of the claims 1–5 is a quantified claim—“For every interpretation...”. So what you're doing in Approach A is making a series of logical inferences involving entire quantified claims. Each of these inferences is perfectly correct, don't get me wrong. But they are somewhat complex inferences. And as the problems get harder, Approach A will become increasingly difficult—it will be hard to understand what the complex quantified statements say, and hard to figure out which other complex quantified statements they imply.

In Approach B, on the other hand, the steps i–vi in the reasoning are not quantified statements about all interpretations of certain sorts; rather, they are assertions about a particular (arbitrarily chosen) interpretation I . Thus the inferences in Approach B are simpler. The reason that these inferences *can* be simpler is that Approach B begins by stating the claim that is to be proved (“Goal”), and then—since that claim is a claim about every interpretation of a certain sort—beginning an argument for that claim by introducing an arbitrary interpretation, and assuming something about it (namely, assumption i).

So my general suggestion is to move away from Approach A and toward Approach B. Get used to approaching proofs “backwards”. Rather than beginning by drawing inferences from the premises you are given, instead, think first about the logical form of the claim you're trying to prove, and determine what arbitrary objects or assumptions (say, for conditional proof or *reductio*) you should introduce, in order to prove that claim. Only then, once the arbitrary

objects and assumptions have been introduced, should you turn to the premises you're given, and apply them to the arbitrary objects.

(But no need to be fanatical about this. You might notice that Approach B contains two parts that are Approach A-like, namely, the inference from i to ii, and the inference from i to iv. Taking Approach B to these two inferences would be tedious. For example, the Approach B argument from i to ii looks like this: "In order to establish ii, let γ be an arbitrary member of Γ . By the definition of set-union, γ is also a member of $\Gamma \cup \Delta$, and so, by i, γ is true in I . Since γ was arbitrarily chosen, ii follows.)

2.

In proofs it's important to use variables correctly. Whenever you use a variable, say, ' x ', you must always be able to answer the question "what is x ?"

For example, suppose you make this claim:

"For every interpretation, I , if P is true in I then so is $P \vee Q$ "

If I ask you "what is I ?", you can answer: it's a bound variable in a quantified claim. (It's like the ' n ' in the claim "for every number n , if n is even then so is $n + 2$ ". In formal predicate logic, it's like the ' x 's in $\forall x(Fx \rightarrow Gx)$.)

Or suppose you wanted to argue for the claim that "For every interpretation, I , if P is true in I then so is $P \vee Q$ ". You might argue as follows:

Let J be an arbitrary interpretation in which P is true. Then, by the t.c. for $\vee$, $P \vee Q$ is true in J .

If I ask you, after the second sentence in this argument, "what is J ?", you can answer: it's an arbitrary interpretation, which I introduced in the first sentence.

But suppose you give a proof whose first line is this:

Let I be an interpretation in which ϕ is true.

This has two variables, ' I ' and ' ϕ '. If I ask you "what is I ?", you can answer: it's an arbitrary interpretation that I'm introducing. But if I ask you "what is ϕ ?", you'll have no answer (unless it was introduced in the statement of the claim you are trying to prove). It wasn't introduced as an arbitrary formula by any earlier lines in the proof (since this is the first line of the proof). And there is no quantified phrase that the variable belongs to (ϕ is like the variable x in the predicate logic formula Fx).

For another example, suppose you are arguing as follows:

“For every interpretation I in which P is true, $P \vee Q$ is also true. And so, $\sim\sim(P \vee Q)$ is also true in I .”

If I ask you after the *first* sentence “what is I ?”, you can answer: it’s a bound variable in a quantified claim. But if I ask you after the *second* sentence “what is I ?”, you can’t answer. ‘ I ’ in the second sentence isn’t a bound variable, because the second sentence doesn’t contain any quantifier phrase (like ‘for any interpretation’ or ‘for some interpretation’), nor has I been introduced as an arbitrary object. It’s as if you wrote, in the formal language of predicate logic: $\forall x(Fx \rightarrow Gx) \wedge Hx$. The first three occurrences of ‘ x ’ are bound variables, but the fourth occurrence—the one in ‘ Hx ’ is a free variable. It doesn’t belong to the quantifier $\forall x$ because the parentheses belong to that quantifier are closed.

Caveat: mathematical prose often leaves the status of variables implicit. For example, one might write this:

1. If a number m is even, then $m = 2n$, for some n (since by definition an even number is one that is the result of multiplying some number by 2).
2. But then $m + 2 = 2n + 2 = 2(n + 1)$.
3. So $m + 2$ is even as well (since it’s the result of multiplying some number, namely, $n + 1$, by 2).
4. Thus for any even number, the result of adding 2 to that number is also even.

At first glance, this proof contains a number of variables that seem neither to have been introduced for arbitrary objects nor do they belong to quantifiers. For example, premise 1 seems to have the form “for any m , if m is even then there exists some n such that $m = 2n$ ”; but then, the ‘ m ’s in lines 2 and 3 don’t seem to belong to quantifiers. But actually, what the phrase “If a number m is even, then...” at the beginning of line 1 really means is “Let m be an arbitrary even number”, and the subsequent ‘ m ’s in the proof are just referring back to this arbitrary even number. For another example, it at first doesn’t seem clear what the ‘ n ’s in lines 2 and 3 refer to, since ‘ n ’ in line 1 is part of the quantified claim “for some n , $m = 2n$ ”. (This has the form $\exists x(\dots)$; the parenthesis for the existential quantifier $\exists x$ are closed.) However, what line 2 really means is this: “Let n be some arbitrary number such that $m = 2n$. (We know from line 1 that there exists some such number.) Then $m + 2 = 2n + 2 = 2(n + 1)$.”

So the moral is: make sure you're clear about the status of your variables! Early on, it may be useful to be very explicit about quantifiers and introductions of arbitrary objects. As you become more fluent in mathematical prose, you may eventually start to leave this implicit, but be sure that your use of variables is always capable of being cleaned up.