Hilbert (and Intuitionism)

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Philosophy of Mathematics

1. Hilbert's program

Rough idea: *truth for the theory of strings*, *deductivism for everything else*

Strings are knowable. They're finite; no threat of paradoxes.

The rest of mathematics can be vindicated without assuming that infinities exist. Hilbert called for mathematicians to:

1. *Formalize* each mathematical theory.
2. *Axiomatize* the theory.
3. Show that the axioms are *complete*
4. Show that the axioms are *consistent*.

Then we can carry on as before, exploring consequences of axioms, without worrying about the paradoxes.

2. Term formalism for arithmetic

Actually, Hilbert accepted truth for arithmetic too, since he was a term formalist about natural numbers; they are strings composed exclusively of vertical strokes: '|', '||', '|||', …

3. Finitism

Actually, Hilbert didn't accept *all* of the theory of strings (and natural numbers); he only accepted "finitary" statements, concepts, and forms of reasoning. These are free of threat of paradox; must be capable of being carried out by a finite being performing mechanical calculations concerning finite objects.

Examples of *finitary* statements: statements about addition, multiplication, and exponentiation involving particular numerals, no matter how large

Examples of *non-finitary* statements: existentially quantified statements about numbers—"there exists some number with feature F". Such statements are about the entire infinite collection of natural numbers; finite beings can't survey all the numbers.

Hypothetical judgments are also finitary, e.g.:

(A) $1 + n = n + 1$

(A) is like a universally quantified claim; it's a recipe for constructing a true claim, whenever the variable is replaced by some particular numeral.

4. Consistency proofs

Hilbert can't establish consistency using models, because the models are infinite. Instead, he aimed to show claims of this form:

> For every finite sequence of finite strings, if the sequence is a legal proof from the axioms, then every line in the sequence fails to be a contradiction

This is finitary because the concepts in it are finitary. Proofs are finite objects. Can check mechanically whether a sequence of strings is a legal proof. It depends only on the "shapes" of the strings in the sequence. And being a contradiction is just a matter of having the shape "A & $\sim A$".

Actually, what is finitary are hypothetical judgments of this form:

(C) If finite sequence S of strings is a legal proof from the axioms, then line n fails to be a contradiction

Hilbert's goal was to give finitary proofs of conclusions of the form (C) for each branch of mathematics, including arithmetic itself, calculus, and set theory.

5. Example consistency proof: propositional logic

The *language*: propositional logic (sentence letters 'P', 'Q', 'R', etc.; the connectives $\sim$ and $\rightarrow$).

The *axioms*:

$$A \to (B \to A)$$
$$\big(A \to (B \to C)\big) \to \big((A \to B) \to (A \to C)\big)$$
$$(\sim A \to \sim B) \to \big((A \to \sim B) \to B\big)$$

The *rules of inference*: there is only one, modus ponens, which allows you to derive C from A together with $A \to C$.

A *contradiction* is any sentence of the form $\sim(A \to A)$. (Equivalent to "A & $\sim A$".)

We are considering proofs *from the logical axioms alone* (thus no premises). These are sequences of formulas each of which is either i) one of the three types of logical axioms, or ii) follows from earlier lines in the sequence by modus ponens.

Proof that no line in any such proof is a contradiction:

Step 1: Define a *truth-value-assignment* as a way of associating either 1 ("true") or 2 ("false") with each sentence letter.

Step 2: Define rules for assigning truth values to complex formulas:

Rule for $\sim$: $\sim A$ is 1 if A is 2, and is 2 if A is 1

Rule for $\to$: $A \to B$ is 1 if A is 2 or B is 1, and is 2 if A is 1 and B is 2

Step 3: Prove that that every axiom is true (i.e., 1) in every truth-value assignment.

Step 4: Prove that if the premises of modus ponens are both true (i.e., 1), so is the conclusion.

Step 5: Use steps 3 and 4 to show that every line in every proof from the logical axioms alone is always true (i.e., 1).

Step 6: use step 5 to show that there exists no such proof of $\sim(A \to A)$.

This reasoning is finitary. Hilbert hoped to do the same for arithmetic, calculus, set theory, etc.

6. Intuitionism

The Dutch mathematician L. E. J. Brouwer thought that mathematical objects and mathematical truths are produced by the mind. He opposed completed infinities and nonconstructive proofs, and as a result called for radical revision of mathematics and even logic.

Example: the decimal expansion of π:

> It isn't an actually infinite object. Rather, more and more of its digits come into existence, as we compute more and more of it.
>
> Questions about the uncomputed portions needn't always have answers.
>
> The proof that some digit occurs infinitely often was nonconstructive; thus intuitionists reject it.
>
> They also reject the claim that either 6 occurs infinitely often in the decimal expansion or it doesn't; thus they reject the "law of the excluded middle" ("either *A* or not-*A*"). They reject classical logic.

Another example: intuitionists reject this proof:

Proof that for some irrational numbers x and y, x^y is rational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we can let $x = y = \sqrt{2}$. And if it is *ir*rational then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ (since $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational). So, since $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational:

(*) Either $\sqrt{2}^{\sqrt{2}}$ is a rational number resulting from raising an irrational number to an irrational power, or else $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is a rational number resulting from raising an irrational number to an irrational power.

Either way, the conclusion follows. □

This is nonconstructive because the proof of statement (*), which is a disjunction, does not include a proof of either disjunct. Intuitionists think the proof makes a logical mistake, in its assumption that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

Intuitionists also think the proof that some digit occurs infinitely many times in the decimal expansion of π makes a logical mistake, at the very end, when it moves from the correct premise (which was established by reductio) that it is *not not* the case that some digit occurs infinitely many times, to the conclusion that some digit occurs infinitely many times. Thus they reject the law of "double negation elimination": that $\sim\sim A$ implies A.

Thus intuitionists reject many standard claims about arithmetic (and also accept some claims about real numbers that ordinary mathematicians reject, such as the claim that every function from real numbers to real numbers is everywhere continuous).

Why does the idea that the mind produces mathematical truths lead to denying the law of the excluded middle? Is the argument this?:

> Since the mind produces mathematical truths, for any mathematical statement A, the following holds:
>
> (*) A if and only if 'A' has been proven
>
> But neither 'the decimal expansion of π contains infinitely many 6s' nor its negation has been proven; so by (*), neither is true; thus 'either the decimal expansion of π contains infinitely many 6s, or it's not the case that the decimal expansion of π contains infinitely many 6s', isn't true either.

No; intuitionists can't accept principle (*). Since 'the decimal expansion of π contains infinitely many 6s', hasn't been proven, (*) tells us that:

> It's not the case that the decimal expansion of π contains infinitely many 6s

And since 'it's not the case that the decimal expansion of π contains infinitely many 6s' hasn't been proven, (*) tells us that:

> It's not the case that it's not the case that the decimal expansion of π contains infinitely many 6s

Thus (*) would lead to contradictions.