Mill

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Philosophy of Mathematics

1. Mill's empiricism

All knowledge of "real" (= "nonverbal") statements comes from the senses. Mathematics (and much of logic) is real, not verbal.

2. "Early and constant experience"

It only appears a priori because it comes from early and constant experience.

3. Geometry: limit concepts

Challenge: observation doesn't really seem to accord with geometry. Observed lines aren't really straight, they always have some thickness, etc.

Mill: the points, lines, circles, etc., that geometry talks about are "limit concepts". He writes as though this means:

> For any chosen degree of accuracy, there exist (or can exist) geometric objects of which the laws of geometry are approximately true to the chosen degree.

Problem 1: maybe there are physical limits (ultimate bits of matter; discrete physical space)

Problem 2: our evidence doesn't really support there being *no* limits to precision

Possible response: we have…

i) *excellent* evidence that we can construct geometric objects with *low* precision

ii) *good* evidence that we can construct geometric objects with *medium* precision

iii) *decent* evidence that we can construct geometric objects with *high* precision

etc.

4. Arithmetic: addition and aggregates

The idea that arithmetic is a priori is due to treating it overly abstractly:

> All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse. (*System of Logic*, VII p. 254)

"There are ten beatings of the pulse" = there is an *aggregate* of pulse-beatings; and it *is a ten*.

Mill suggests that '$2+1=3$' conveys that:

> [aggregates] of objects exist, which while they impress the senses thus, $^{\circ}{\circ}^{\circ}$, may be separated into two parts, thus, $\circ\circ\quad\circ$" (p. 257).

Frege replied:

> What a mercy, then, that not everything in the world is nailed down; for if it were, we should not be able to bring off this separation, and $2+1$ would not be 3! What a pity that Mill did not also illustrate the physical facts underlying the numbers 0 and 1! (*Foundations of Arithmetic*, p. 9)

But Mill really means something like this:

> If the aggregate A of circles on the top of the diagram is a two and the aggregate B of circles on the bottom of the diagram is a one, $A \cup B$ is a three

where "$A \cup B$" means "the aggregate composed of the (disjoint) aggregates A and B". Though actually number talk is more abstract than that:

> All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse. But though numbers must be numbers of something, they may be numbers of anything. Propositions, therefore, concerning numbers, have the remarkable peculiarity that they are propositions concerning all things whatsoever, all objects, all existences of every kind, known to our experience. (pp. 254–5)

So really his view is that '$2+1=3$' means:

> For any aggregate X that is a two and any aggregate Y that is a one, $X \cup Y$ is a three

5. The problem of large sums

Problem: We can be justified in believing '$1457 + 235 = 1692$' only if we have counted many large aggregates of these sizes in the past.

6. Definitions of counting numerals

Mill's response: base the knowledge on general principles that are known through experience with smaller aggregates.

> "Aggregate A is a four" means that $A = B \cup C$, for some aggregate B which is a one, and some aggregate C which is a three
>
> "Aggregate A is a three" means that $A = B \cup C$, for some aggregate B which is a one, and some aggregate C which is a two
>
> "Aggregate A is a two" means that $A = B \cup C$, for some aggregate B which is a one, and some aggregate C which is a one

In order to justify '$5 + 3 = 8$', i.e.,:

> For any aggregate X that is a five and any aggregate Y that is a three, $X \cup Y$ is an eight

We can make this argument:

> Let X be a five Y be a three. By definition of 'is a three',
>
> $$Y = A \cup B$$
>
> for some A and B where A is a one and B is a two. By definition of 'is a two',
>
> $$B = C \cup D$$
>
> for some C and D, each of which is a one. Thus:
>
> $$X \cup Y = X \cup (A \cup B) = X \cup \big(A \cup (C \cup D)\big)$$
>
> Assuming that $\cup$ is associative:
>
> $$X \cup \big(A \cup (C \cup D)\big) = (X \cup A) \cup (C \cup D) = \big((X \cup A) \cup C\big) \cup D$$
>
> By definition of 'is a six', $X \cup A$ is a six. By definition of 'is a seven', $(X \cup A) \cup C$ is a seven. By definition of 'is an eight', $((X \cup A) \cup C) \cup D$ is an eight. Thus $X \cup Y$ is an eight.

The argument rests on associativity (and other general principles) which Mill can claim are known by our experiences with small aggregates.

7. The epistemology of aggregates

Does experience really justify us in believing that aggregates exist?

Also, Frege said:

> . . . there are very various manners in which an [aggregate] can be separated into parts, and we cannot say that one alone would be characteristic. For example, a bundle of straw can be separated into parts by cutting all the straws in half, or by splitting it up into single straws, or by dividing it into two bundles. (*Foundations of Arithmetic*, p. 30)

Mill responded that the following are all distinct aggregates:

> the aggregate of the straws (which is a ten)
>
> the aggregate of the straw halves (which is a twenty)
>
> the aggregate of the smaller bundles (which is a two)

This makes the epistemology of aggregates even harder.

8. Inadequate scope

What would Mill say about multiplication; e.g., '$2 \times 5 = 10$'? Perhaps:

> For any aggregate of aggregates, X, if X is a two, and if each part of X is a five, and if no two parts of X and no two parts-of-parts of X overlap, then the aggregate of parts-of-parts of X is a ten.

But now he is appealing to aggregates of aggregates, which exacerbates the epistemic problems. (Which will get even worse, if he needs aggregates of aggregates of aggregates to account for exponentiation.)

Worse, Mill has no account of how we know *general* sentences of arithmetic, or even what they mean, such as 'for any numbers m and n, $m + n = n + m$' and 'for any number m, there is a larger number, n, that is prime'.

9. Abstract mathematics

Really, the central problem with Mill is that his account of mathematics is too tied to concrete procedures such as counting. Some of mathematics is indeed concrete in this way: the natural numbers are tied to counting, and the real numbers to measurement. But modern mathematics is far more abstract than that. What concrete procedures are functions, or arbitrary groups, etc. tied to?