# ZF Set Theory

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Philosophy of Mathematics

Mathematicians didn't give up on infinity. They developed *ZF set theory*, an (apparently) consistent theory of sets that can provide a foundation for all of mathematics.

**Set-theoretic platonism**

Mathematics is about the realm of sets, which exist independently of us. All branches of mathematics reduce to set theory; all mathematical entities are sets.

## 1. The ZF Solution to Russell's Paradox

*Russell's set R doesn't exist.*

Thus the cavalier attitude toward sets needs to go; we must reject:

**Naïve Comprehension**

For any "condition", there exists a corresponding set: the set of all and only those things that satisfy the condition.

## 2. The ZF axioms

But we still need *something like* Naïve Comprehension, so we know that we can define the sets we need in mathematics. The ZF strategy for doing so: first formulate "expansion" axioms, saying that certain sets exist:

**“Expansion” axioms**

*Null set:* There exists a set $\varnothing$ containing no members

*Pairing:* For any sets $a$ and $b$, there exists a set containing just $a$ and $b$ (i.e.: $\{a, b\}$)

*Unions:* For any sets $A$ and $B$, there exists a set, $A \cup B$, containing all the members of $A$ and also all the members of $B$

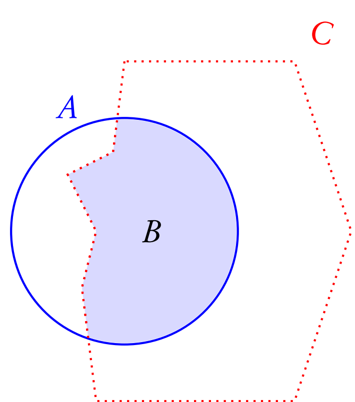
*Infinity:* There exists a set, $A$, that contains the null set, and is such that for any $a$ that is a member of $A$, the set $a \cup \{a\}$ is also a member of $A$. (Any such set $A$ must be infinite, since it contains all of these sets: $\varnothing, \{\varnothing\}, \{\{\varnothing\}, \varnothing\}, \ldots$.)

*Power set:* For any set, $A$, the power set of $A$ (i.e., the set of $A$'s subsets) also exists

Second, formulate a “contraction” axiom, which lets us use arbitrary conditions to pick out *subsets* of given sets:

**Axiom of separation**

Suppose some set $A$ exists, and let $C$ be any condition (i.e., any formula in the language of set theory). Then there exists a set $B$ consisting of all and only the members of $A$ that satisfy condition $C$.



For any set $A$, Separation says that there exists a set $B$ of all non-self-membered-sets *that are in* $A$ (we can let $C$ be "is not a member of itself"). Now, the assumption that Russell's set $R$ isn't a member of $R$ led to contradiction:

i) Suppose that $R$ is not a member of $R$. So it doesn't contain itself.

ii) But $R$ contains *every set that doesn't contain itself*.

iii) So $R$ must be a member of $R$ after all.

But the assumption that $B$ isn't a member of $B$ doesn't lead to a contradiction:

i) Suppose that $B$ is not a member of $B$. So it isn't a member of itself.

ii) But $B$ contains *every set that doesn't contain itself* <u>*and which is a member of* $A$</u>.

iii) ??

## 3. Reducing mathematical entities to sets

"Branch $B$ of mathematics can be reduced to ZF" means: when you define the primitive expressions of $B$ in set-theoretic terms, the axioms of $B$ become theorems of ZF.

ZF can reduce all branches of mathematics because the expansion axioms provide enough "raw materials" that may be "cut to size" using the axiom of Separation, to provide entities with any structural features we like.

### 3.1 Ordered pairs

Challenge to this idea: this ZF axiom implies that sets are unordered:

**Axiom of Extensionality**

If sets $A$ and $B$ have exactly the same members, then $A = B$

For instance, $\{a, b\} = \{b, a\}$. But we can construct entities for which order *is* significant:

**Definition of ordered pairs**

The ordered pair $\langle a, b\rangle$ is defined as the set $\{\{a\}, \{a, b\}\}$.

This definition implies that $\langle a, b\rangle \neq \langle b, a\rangle$, and in general that:

> Ordered pairs $\langle a, b\rangle$ and $\langle c, d\rangle$ are identical if and only if $a = c$ and $b = d$.

### 3.2 Functions

**Definition of functions**

A function is a set of ordered pairs such that whenever $\langle a, b\rangle$ and $\langle a, c\rangle$ are in the set, $b = c$

If $f$ is a function, "$f(a)$" can be defined as the unique $b$ such that $\langle a, b\rangle$ is a member of $f$ (if no such $b$, $f(a)$ is undefined).

### 3.3 Natural numbers

**Definition of natural numbers**

$0$ is defined as $\varnothing$ (the empty set)

For any $m$, the *successor* of $m$ is defined as the set $m \cup \{m\}$

Thus $0 = \varnothing$, $1 = \{\varnothing\}$, $2 = \{\varnothing, \{\varnothing\}\}$, etc.

### 3.4 Reduction of rational numbers

Can we define $\frac{m}{n}$ as the ordered pair $\langle m, n\rangle$? Not quite, because, e.g., $\frac{1}{2} = \frac{2}{4}$. Instead, we define rational numbers as "equivalence classes" of ordered pairs. For example:

$$\frac{1}{2} = \left\{\langle 1, 2\rangle, \langle 2, 4\rangle, \langle 3, 6\rangle, \ldots\right\}$$

### 3.5 Real numbers

A real number can be defined, roughly, as the set of all rational numbers that are less than that real number. Non-circular version of this idea:

**Definition of real numbers**

A real number is defined as any set, $A$, of rational numbers that:

1. is nonempty and is not the set of all rational numbers
2. is downward-closed (i.e., if some rational number is a member of $A$, then every smaller rational number is also a member of $A$)
3. has no largest member (i.e., for any member of $A$ there is a larger member of $A$)

### 3.6 Groups

Arbitrary "algebraic structures" can be defined as a set (containing the "numbers" in question), and one or more functions ("operations") giving their structure. Example:

**Set-theoretic definition of a *group***

A group is an ordered pair $\langle A, * \rangle$ such that:

1. $A$ is a nonempty set
2. $*$ is a function that maps any ordered pair of members of $A$ to another member of $A$. (We'll abbreviate "$*(\langle x, y \rangle)$" as "$x * y$".)
3. $(x * y) * z = x * (y * z)$, for any $x, y, z \in A$ (Associativity)
4. There exists an $e \in A$ such that for any $x \in A$, $x * e = e * x = x$ (Identity element)
5. For any $x \in A$ there exists some $y \in A$ such that $x * y = y * x = e$ (Inverses)

### 3.7 Geometry

Example of set-theoretic definitions in geometry:

**Set-theoretic definition of a metric space**

A metric space is an ordered pair $\langle S, d \rangle$ where:

1. $S$ is a set
2. $d$ is a two-place function mapping any pair of members of $S$ to a real number
3. $d(x,x) = 0$, for any $x \in S$
4. $d(x,y) > 0$ if $x \neq y$, for any $x, y \in S$
5. $d(x,y) = d(y,x)$, for any $x, y \in S$
6. $d(x,y) + d(y,z) \geq d(x,z)$, for any $x, y, z \in S$

Like all set-theoretic reductions of mathematical objects, this relies on the modern, abstract approach to mathematics. It doesn't matter whether the members of $S$ are "really" points, or that $d$ "really" assigns distances. All that matters is the structure.