The Move to Abstraction

Ted Sider
Philosophy of Mathematics

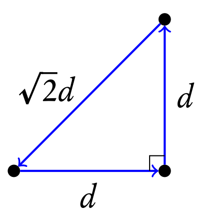
(Gowers chapter 2 is a good resource on this topic)

1. Concrete approaches to numbers

Concrete = tied to applications in the nonmathematical world

Natural numbers: counting

Real numbers: measuring (e.g. length—rationals won't do)



$$d^2 + d^2 = (\sqrt{2}d)^2 = 2d^2$$

(Although the ancient Greeks didn't recognize rational or real numbers, they spoke of *magnitudes* (lengths, angles, areas) in related ways. E.g.:

"the ratio of magnitude a to magnitude b is 3 to 4" means that $4a = 3b$, i.e. $a + a + a + a = b + b + b$, where "+" is addition of the magnitude in question

"the ratio between a and b is the same as the ratio between c and d" means that for any natural numbers m and n:

if $ma < nb$ then $mc < nd$

if $ma = nb$ then $mc = nd$

if $ma > nb$ then $mc > nd$

The latter statement makes sense even if the ratio is irrational.)

2. Move toward abstraction

Mathematics gradually became less tied to concrete applications. Early example (in the Medieval Islamic world): developing rules for finding the solutions to equations, such as $x^2+ax=c$, which called for real numbers (and later, complex numbers).

3. Abstract algebra

In modern algebra, we talk about number systems in which numbers behave differently, such as addition "mod" 5, in which the numbers are just $0, 1, 2, 3, 4$.

In this abstract approach, all that matters is *structure*: how the operation behaves (e.g. whether it's commutative). It doesn't matter what the "numbers" are. They can be anything, e.g., rotations of a geometric object or polynomials, so long as they satisfy the axioms for the structure in question.

We even treat the natural numbers structurally; they are defined by rules like:

$$
\begin{aligned}
m + 0 &= m \\
m \times (n + p) &= (m \times n) + (m \times p) \\
m \times 0 &= 0 \\
m \times 1 &= m
\end{aligned}
$$

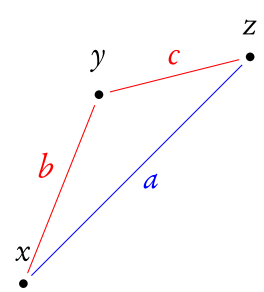
4. Abstract geometry

Mathematical spaces are defined by their structure, which we stipulate. “Euclidean space” obeys Euclid’s axioms; “Hyperbolic space” obeys different axioms.

(Mathematical spaces are therefore distinct from *physical space*, which is discovered, not stipulated.)

Example: a *metric space* is some chosen “points”, together with some assignment of real numbers to each pair of points (“distances”), obeying these rules:

1. the distance between a point and itself is always 0
2. the distance between a point and any other point is always greater than 1
3. the distance from one point to another is always the same as the distance from the second point to the first
4. the distance from point x to point y plus the distance from point y to point z is always greater than or equal to the distance from point x to point z



$$b + c \geq a$$

The “points” and “distances” can be anything (fish, bananas), so long as they satisfy these conditions.

5. Abstraction and Mill