Conventionalism and Intuitionism

Ted Sider
Philosophy of Mathematics

1. Conventionalism

1.1 Truth in virtue of meaning

"Mathematical (and logical) claims are true simply because of what they mean."

1.2 Ayer on analyticity

Trying to improve on Kant's definition of analyticity ("the subject is contained in the predicate")

> a proposition is analytic when its validity depends solely on the definitions of the symbols it contains… If one knows what is the function of the words "either," "or," and "not," one can see that any proposition of the form "Either *p* is true or *p* is not true" is valid, independently of experience. (?, pp. 78-9)

But this suggests two different conceptions of analyticity.

1.3 "If one knows… then one can see…"

This suggests: *a sentence is analytic if and only if anyone who understands that sentence can know that it is true*.

But then a sentence's being analytic couldn't *explain* how we know it.

1.4 "depends solely on the definitions…"

This suggests: *a sentence is analytic if and only if its truth is produced by definitions*.

But "definitions" are usually understood as the introduction of *synonyms*, which (as Quine pointed out) don't *create* truths, but merely *transform* truths.

"Taylor Swift is a female sibling" ⇒ "Taylor Swift is a sister"

"All female siblings are sisters" ⇒ "All sisters are siblings"

(via the definition of 'sister' as meaning 'female sibling')

Ayer seems to think that we can define terms by *stipulating the truth of* certain sentences containing them. For example, as part of my definition of the terms 'if' and 'then', I can say:

> "I hereby stipulate that the sentence 'If snow is white then snow is white' is to be true"

Quine: this only produces *one* logical truth; but what about the infinitely many others? Could we use *general* stipulations, such as:

(GS) For every x, if x is a sentence of the form 'If S then S', then x is true

? No: we'll already need to know logic, in order to use (GS) to conclude particular logical truths, such as:

(PLT) 'If grass is green then grass is green' is true

First we'd use (GS) to conclude

> If 'If grass is green then grass is green' is a sentence of the form 'If S then S', then 'If grass is green then grass is green' is true

which requires the logical rule of universal instantiation. Then we'd need to use this plus the further fact that:

> 'If grass is green then grass is green' is a sentence of the form 'If S then S'

to conclude (PLT), which requires using the logical rule of modus ponens.

2. Intuitionism

The Dutch mathematician L. E. J. Brouwer thought that mathematics is produced by the mind. He opposed completed infinities and nonconstructive proofs, and called for radical revision of mathematics and even logic.

Example: the decimal expansion of π:

> It isn't an actually infinite object. Rather, more and more of its digits come into existence, as we compute more and more of it.
>
> Questions about the uncomputed portions needn't always have answers.
>
> The proof that some nonzero digit occurs infinitely often was nonconstructive; thus intuitionists reject it.

> They also reject the claim that either 6 occurs infinitely often in the decimal expansion or it doesn't; thus they reject the "law of the excluded middle" ("either *A* or not-*A*"). They reject classical logic.

For another example, consider the statement:

> (*) Either $\sqrt{2}^{\sqrt{2}}$ is a rational number resulting from raising an irrational number to an irrational power, or else $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is a rational number resulting from raising an irrational number to an irrational power.

Intuitionists reject the following proof of (*):

> Either $\sqrt{2}^{\sqrt{2}}$ is rational, or it is isn't. If it's rational, then since $\sqrt{2}$ is *ir*rational, the left hand disjunct of (*) (that is, the left-hand part of the overall 'or' statement) is true, and so (*) itself is true. And if $\sqrt{2}^{\sqrt{2}}$ isn't rational then since $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$, which is rational, the right-hand disjunct of (*) is true, and so again, (*) itself is true. Either way, (*) is true.

This is nonconstructive because the proof of statement (*), which is a disjunction, does not include a proof of either disjunct. Intuitionists think the proof makes a logical mistake, in its assumption that $\sqrt{2}^{\sqrt{2}}$ is either rational or is not rational.

Intuitionists also think the proof that some digit occurs infinitely many times in the decimal expansion of π makes a logical mistake, at the very end, when it moves from the correct premise (which was established by reductio) that it is *not not* the case that some digit occurs infinitely many times, to the conclusion that some digit occurs infinitely many times. Thus they reject the law of "double negation elimination": that $\sim\sim A$ implies A.

Thus intuitionists reject many standard claims about arithmetic (and also accept some claims about real numbers that ordinary mathematicians reject, such as the claim that every function from real numbers to real numbers is everywhere continuous).

Why does the idea that the mind produces mathematical truths lead to denying the law of the excluded middle? Is the argument this?:

> Since the mind produces mathematical truths, for any mathematical statement A, the following holds:
>
> (*) A if and only if 'A' has been proven
>
> But neither 'the decimal expansion of π contains infinitely many 6s' nor its negation has been proven; so by (*), neither is true; thus 'either the decimal expansion of π contains infinitely many 6s, or it's not the case that the decimal expansion of π contains infinitely many 6s', isn't true either.

No; intuitionists can't accept principle (*). Since 'the decimal expansion of π contains infinitely many 6s', hasn't been proven, (*) tells us that:

> It's not the case that the decimal expansion of π contains infinitely many 6s

And since 'it's not the case that the decimal expansion of π contains infinitely many 6s' hasn't been proven, (*) tells us that:

> It's not the case that it's not the case that the decimal expansion of π contains infinitely many 6s

Thus (*) would lead to contradictions.