# Intuitionism

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Mathematical objects and mathematical truths result from the activity of the mind. This leads to radical revisions of standard mathematics and logic.

## 1. Mind dependence

Human mathematical activity somehow produces the truths of mathematics.

### 1.1 No "completed" infinities

In part this means rejecting "completed infinities". To date we have constructed around 30 trillion digits of $\pi$'s decimal expansion. Intuitionists reject the platonist's idea that the infinitely many remaining digits "exist already".

### 1.2 Rejection of LEM

Call a "diabolical sequence" a sequence of 666 consecutive 6s. According to standard logic, this is a logical truth:

(D) Either some diabolical sequence occurs in the decimal expansion of $\pi$, or it's not the case that some diabolical sequence occurs in the decimal expansion of $\pi$

Intuitionists reject this. Since the infinite decimal expansion of $\pi$ doesn't "already exist", until we either observe a diabolical sequence, or prove that no such sequence can appear, we cannot assume that (D) is true. Thus intuitionists reject:

**Law of the excluded middle (LEM)** $\quad A$ or not-$A$

### 1.3 Constructivism

Intuitionists also insist that mathematical proofs be *constructive*—provide an explicit construction or computation.

### 1.3.1 Examples of constructive proofs

*Proofs of particular mathematical facts, such as that* $2 \times 13 = 26$.

*Proof that the number* 26 *is even.* The definition of 'even' is this: $n$ is even iff for some $m$, $n = 2 \cdot m$. Thus what we are trying to prove is:

$$\text{for some } m,\ 26 = 2 \cdot m$$

Now, we can verify by direct computation that $26 = 2 \cdot 13$. Thus there does exist some such $m$—namely, 13. □

(A constructive proof of an existentially quantified statement consists of proving some particular instance.)

*Proof that for every even number* $n$, $n+2$ *is even.* Consider any even number, $n$. Since $n$ is even, $n = 2 \cdot m$, for some $m$. Thus $n+2 = 2 \cdot m + 2 = 2(m+1)$. Thus there is some number $p$ such that $n+2 = 2 \cdot p$—namely, $m+1$. Therefore $n+2$ is even. □

(A constructive proof of a universal generalization consists of a specification of a *method* for constructing a proof of any given instance.)

### 1.3.2 Examples of nonconstructive proofs

*Proof that there exist irrational numbers x and y such that* $x^y$ *is rational.* Consider $\sqrt{2}^{\sqrt{2}}$. Either it is rational or it is irrational. If it is rational then we are done: let $x = \sqrt{2}$ and $y = \sqrt{2}$. ($\sqrt{2}$ is known to be irrational.) Otherwise let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. For then $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$, which is rational. □

*Proof that some nonzero digit occurs infinitely often in the decimal expansion of* $\pi$. Suppose that each of the digits 1–9 occurs only finitely many times. Then once all those digits are done occurring, from that point onward the decimal expansion would consist only of `0`s. $\pi$ would therefore be a rational number; but it is known to be irrational; contradiction. □

In each case, an existential statement is proved but no instance is proved.

# 2. Revision of logic

## 2.1 Double-negation elimination

In addition to rejecting LEM, intuitionists reject all principles that would lead to LEM, such as:

**Double negation elimination (DNE)** If not-not-$A$ then $A$

Argument from DNE to LEM:

| | | |
|---|---|---|
| 1. | ~~show~~ $\sim\sim(A \vee \sim A)$ | Reductio |
| 2. | $\sim(A \vee \sim A)$ | Assume |
| 3. | ~~show~~ $\sim A$ | Reductio |
| 4. | $A$ | Assume |
| 5. | $A \vee \sim A$ | 4, $\vee$-intro |
| 6. | Contradiction | 2, 5 |
| 7. | $A \vee \sim A$ | 3, $\vee$-intro |
| 8. | Contradiction | 2,7 |
| 9. | $A \vee \sim A$ | 1, DNE |

This uses these rules, which intuitionists accept:

$$\frac{A}{A \vee B} \qquad \frac{A}{B \vee A} \qquad \vee\text{-intro}$$

$$\frac{A \Rightarrow \text{ a contradiction}}{\sim A} \qquad \text{Reductio}$$

*Note:* intuitionists reject the following rule:

$$\frac{\sim A \Rightarrow \text{ a contradiction}}{A} \qquad \text{Strong reductio}$$

For it would lead to DNE:

| | | |
|---|---|---|
| 1. | $\sim\sim A$ | Suppose |
| 2. | ~~show~~ $A$ | Strong reductio |
| 3. | $\sim A$ | Assume |
| 4. | Contradiction | 2, 3 |

### 2.2 Existentials and universals

Here is another standard inference rule that intuitionists reject:

$$\frac{\sim\forall x F}{\exists x \sim F}$$

You can prove $\sim\forall x F$ by showing that $\forall x F$ leads to a contradiction. But to prove $\exists x \sim F$ you need something more: a proof of some particular instance of this existential.

Example: "every nonzero digit occurs only finitely many times in the decimal expansion of $\pi$" leads to a contradiction. So by reductio, not every nonzero digit appears only finitely many times. But intuitionists will not conclude that: Some nonzero digit is such that it does not appear only finitely many times.

(In standard logic you can derive $\exists x \sim F x$ from $\sim\forall x F x$:

| | | |
|---|---|---|
| 1. | $\sim\forall x Fx$ | Suppose |
| 2. | ~~show~~ $\exists x{\sim}Fx$ | Strong reductio |
| 3. | $\sim\exists x{\sim}Fx$ | Assume |
| 4. | ~~show~~ $\forall x Fx$ | Universal proof |
| 5. | ~~show~~ $Fa$ | Strong Reductio |
| 6. | $\sim Fa$ | Assume |
| 7. | $\exists x{\sim}Fx$ | 6, ∃-intro |
| 8. | Contradiction | 3, 7 |
| 9. | Contradiction | 1, 4 |

Fortunately for intuitionists, this proof uses Strong reductio, which they reject.)

### 2.3 Rejection and denial

Consider this instance of LEM:

(1) $\sqrt{2}^{\sqrt{2}}$ is rational or $\sqrt{2}^{\sqrt{2}}$ is not rational

The first part of the argument from DNE to LEM shows that the negation of (any instance of) LEM leads to a contradiction. Thus intuitionists will not accept the negation of (1). Rather, they will *refrain from accepting* (1), and they will say that it is not a logical truth.

## 3. A puzzle about intuitionism

It isn't clear why the belief that mathematics is mind-dependent should lead to denying LEM.

*Attempt 1*: Assume the…

**Flat-footed truth-proof connection** For any mathematical statement, $A$: $A$ if and only if it has been proven that $A$

…and then argue as follows:

> Let $S$ be the statement that $\sqrt{2}^{\sqrt{2}}$ is rational. We haven't proven $S$, and we haven't proven $\sim S$. So by the flat-footed truth-proof connection, neither $S$ nor $\sim S$ is true, and so $S \vee \sim S$ isn't true.

Problem: since we have not proven $S$, and haven't proven $\sim S$, the flat-footed truth-proof connection would imply $\sim\sim S$, which is a contradiction.

*Attempt 2:* Assume the …

**Slightly less flat-footed truth-proof connection** For any mathematical statement, $A$: $A$ if and only if it *can* be proven that $A$

…and argue that

> Neither $S$ nor $\sim S$ *can* be proven. So by the Slightly less flat-footed truth-proof connection, neither $S$ nor $\sim S$ is true, and so $S \vee \sim S$ isn't true.

Problem: the claim that Neither $S$ nor $\sim S$ *can* be proven, plus the Slightly less flat-footed truth-proof connection, still leads to the contradictory claims $\sim S$ and $\sim\sim S$

*Attempt 3:* Continue to assume the Slightly less flat-footed truth-proof connection, and argue as follows:

> It is not a logical truth that (i) "$A$ is provable or $A$ is not provable". But if LEM were a valid rule then $A \vee \sim A$ would be a logical truth after all (since it would follow from (1) by the Slightly less flat-footed truth-proof connection). Therefore LEM is not a valid rule.

Problem: the assumption that (i)—an instance of LEM—isn't a logical truth begs the question.

Sophisticated intuitionists (e.g., Dummett) have developed more complex views. A sketch:

> Intuitionism leads to a distinctive and radical account of meaning itself. We can't think of meaning as a relation to bits of the world, since intuitionists reject this picture for mathematics. Instead, we need to think of the meaning of a sentence as having to do with the method for proving it. (Think of Wittgenstein's slogan that "meaning is use".) And we need to think of the meanings of logical worlds, like $\vee$ and $\sim$, as having to do with the method for proving sentences that contain them.
>
> Given this theory of meaning, we should not speak of truth at all; we should instead speak of assertability. A logical truth is no longer conceived as a statement that is true no matter what, but rather, a statement that is assertable no matter what.
>
> We are entitled to assert a statement if and only if we have a certain kind of "canonical proof" of that statement. A canonical proof of a disjunctive statement $A \vee B$ consists either of a proof of $A$ or a proof of $B$. A canonical proof of a negation $\sim C$ consists of a proof that any proof of $C$ could be transformed into the proof of a contradiction.
>
> Now, if $S \vee \sim S$ were a logical truth, there would need to be a logical guarantee that we are entitled to assert this. Thus we would need a logical guarantee of the existence of one of the two kinds of proofs: either a proof of $S$, or a proof that any proof of $S$ could be transformed into a proof of a contradiction. There is no such logical guarantee; therefore $S \vee \sim S$ is not a logical truth.

## 4. Intuitionism and analysis

Think of real numbers as their decimal expansions. But decimal expansions seem like completed infinities.

Brouwer: sequences are merely potentially infinite. For some, we have a rule that determines new additions. For others, "free choice" sequences, a "creative subject" chooses new additions to the sequence, ungoverned by any rule.

For Brouwer, a number $a$ has a property $P$ only if this can be proven. So if $a$ is based on a free-choice sequence, $a$ has $P$ only if we can establish this by consulting some initial segment of the sequence (since future actions of creative subjects are unknown). So if $a$ has $P$, there must be some interval around $a$ in which all the members have $P$.

This kind of reasoning leads Brouwer to conclusions that contradict standard mathematics, such as that every function from real numbers to real numbers is continuous everywhere.