Modern Logic

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Philosophy of Mathematics

(A good resource: `https://plato.stanford.edu/entries/logic-firstorder-emergence/`)

1. The need for logic

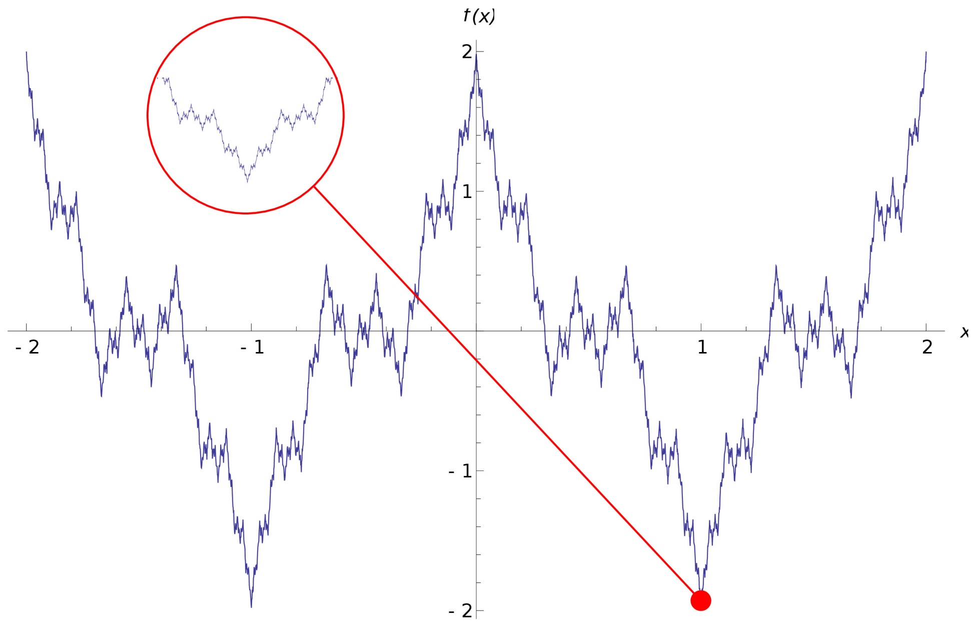
1.1 Abstract mathematics

Given the modern, abstract approach to mathematics, we can no longer be guided by our intuitions; we need to know how to draw out consequences of axioms without any "outside" knowledge sneaking in.

1.2 "Obvious" statements

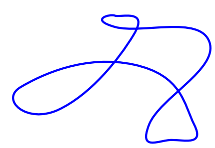
Continuous curves needn't be differentiable (they can have sharp edges):

But it seemed obvious that *a continuous curve is differentiable at all but some isolated points*, until curves were discovered that are continuous but differentiable *nowhere*, e.g., the Weierstrass function:

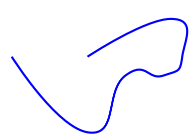


No matter how closely you zoom in, the curve remains "everywhere jagged".

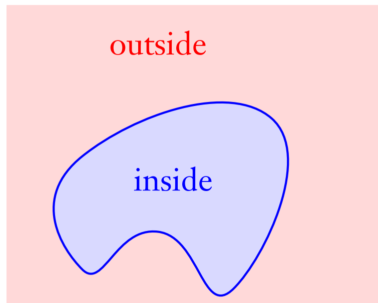
Another example. A "simple closed" curve is one that doesn't cross itself, unlike this one:



and which meets itself at the end, unlike this one:



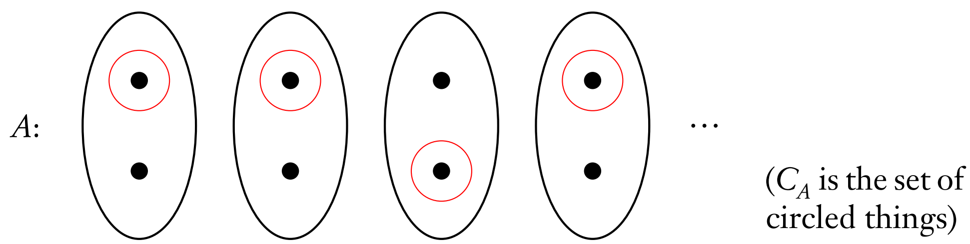
It seems obvious that *any simple closed curve splits the plane into two parts*, *the inside and the outside of the curve*:



It *is* true (it's called the Jordan Curve theorem) but is difficult to prove.

Final example, from set theory:

Axiom of choice If A is a set of nonoverlapping sets, there exists some set C_A containing exactly one member of each member of A

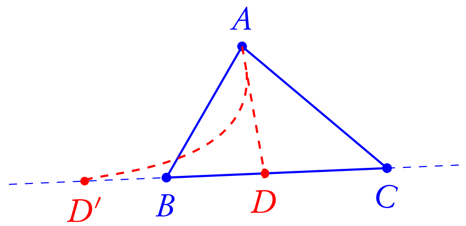


This seems completely obvious (people often assume it without noticing) but it doesn't follow from familiar principles and needs to be added as an axiom.

These examples show that whether something follows from some axioms can be subtle, nonobvious, and difficult.

1.3 Hilbert's geometry

There are certain gaps in Euclid's proofs. E.g., Euclid's axioms don't strictly guarantee that the bisector of an angle of a triangle intersects the opposite side:



In 1899 David Hilbert filled in the gaps by making certain assumptions explicit:

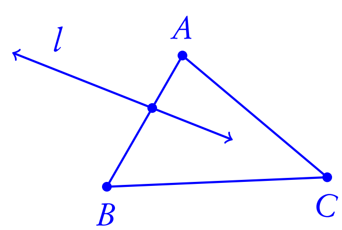
every line contains at least two points.

there exist three noncollinear points

and introducing new primitive notions, e.g., that of one point being *between* two others. New axioms for betweeness included:

Given any three distinct points on a line, one and only one of the three is between the other two.

Pasch's axiom Let A, B, and C be any three noncollinear points, and let l be a line that doesn't contain A, B, or C. If l contains a point between A and B, then it must also contain a point between A and C, or a point between B and C.



Hilbert's goals:

1. Show that the axioms are consistent
2. Show that the axioms are independent

3. Show that the axioms are "complete" in that they imply everything they "should"

This requires a *mathematical* treatment of logic.

2. Modernization of logic

2.1 Aristotle

Aristotle's logic was considered the last word on logic for over 2000 years. He introduced arguments, with premises and conclusions; proofs; and rules of inference.

Aristotle presupposed a certain conception of logical form. A sentence has a subject and a predicate. Subjects can be singular ('Socrates') or general ('All humans'). The predicate can be affirmed or denied of the subject: 'Socrates is mortal' or 'Socrates is not mortal'. Example Aristotelian sentences:

> All humans are mortal
>
> Not all humans are mortal
>
> Some humans are mortal
>
> No humans are mortal

An Aristotelian valid syllogism:

> All Fs are Gs
>
> Some H is F
>
> Therefore, some H is G

An *in*valid syllogism:

> All Fs are Gs
>
> Some H is G
>
> Therefore, some H is F

2.2 Frege

Frege's 1879 *Begriffsschrift* introduced modern predicate logic. The core was innovations in logical form.

Like Aristotle, Frege recognized simple subject-predicate sentences:

> "Socrates is human": Hs
>
> "Frege is a logician": Lf

But Frege allowed sentences with multiple subjects (multi-place predicates):

> "Plato taught Aristotle": Tpa
>
> "a is between b and c": $Babc$

He also introduced *quantifiers* and *variables*:

> "Everything is human": $\forall x Hx$
>
> "Everything taught Aristotle: $\forall x Txa$
>
> "Plato taught everything": $\forall x Tpx$
>
> "Something taught something": $\exists x \exists y Txy$

And he introduced *sentential connectives*:

> "Socrates is human and Frege is a logician": $Hs \,\&\, Lf$
>
> "Socrates is not human": $\sim Hs$
>
> "If Socrates is human then Frege is a logician": $Hs \rightarrow Lf$

Frege can represent all the Aristotelian forms, for example:

> "All Fs are Gs": $\forall x(Fx \rightarrow Gx)$
>
> "Some Fs are Gs": $\exists x(Fx \,\&\, Gx)$
>
> "No Fs are Gs": $\sim\exists x(Fx \,\&\, Gx)$
>
> "Not all Fs are Gs": $\sim\forall x(Fx \rightarrow Gx)$

But Frege's logic was more powerful. Aristotle's best symbolizations of:

(1) Some lawyer is respected by every politician

(2) Every politician respects some lawyer

are

$$\text{Some } \underbrace{\text{lawyers}}_{F\text{s}} \text{ are } \underbrace{\text{things-that-are-respected-by-every-politician}}_{G\text{s}}$$

$$\text{All } \underbrace{\text{politicians}}_{H\text{s}} \text{ are } \underbrace{\text{things-that-respect-some-lawyer}}_{I\text{s}}$$

But (1) logically implies (2), whereas Aristotle's symbolization of (1), "All Fs are Gs", doesn't imply his symbolization of (2), "All Hs are Is", since those symbolizations ignore the logical structure of the predicates 'thing-that-is-respected-by-every-politician' and 'thing-that-respects-some-lawyer'

Frege's representations of (1) and (2) "break up" those sentences further:

$$\exists y(Ly \,\&\, \forall x(Px \to Rxy))$$

$$\forall x(Px \to \exists y(Ly \,\&\, Rxy))$$

Given Frege's logical rules, the second can be derived from the first.

2.3 First- versus second-order logic

First-order logic (the kind in typical textbooks):

$$\forall x Gx \qquad \exists x \exists y Bxy$$

Second-order logic:

$$\exists F Fa \qquad \forall R(Rab \to Rba)$$

In first-order logic, variables must, grammatically, be subjects. In second-order logic, variables that are predicates are also allowed. Frege's logic was second-order.

What do sentences of second-order logic mean? Natural answer:

$\exists F Fa$: "there is some property (one-place concept) that a has"

$\forall R(Rab \to Rba)$: "for every relation (two-place concept) that a bears to b, b bears that same relation to a"

2.4 Mathematical logic

Hilbert in 1917 proposed a *mathematical* investigation of logic. We divide mathematical logic into these branches:

Syntax Definition *grammatical sentences*

Semantics Definition of *interpretation* (e.g., assignments of truth values in propositional logic)

Proof theory Definition of proof. *Axioms* and *rules of inference* are defined. A *proof* is then a finite series of grammatical sentences in which each member of the series is either an axiom or follows from an earlier member of the series by some rule.

One can then prove *mathematical theorems* which are *about logic*; e.g.:

Consistency of propositional logic In a certain axiomatic system for propositional logic, no contradictions (i.e., sentences of the form P & $\sim P$) can be proved

Completeness of propositional logic In this axiomatic system, every tautology (sentence that has all "Ts" in the final column of its truth table) can be proved