NonEuclidean Geometry

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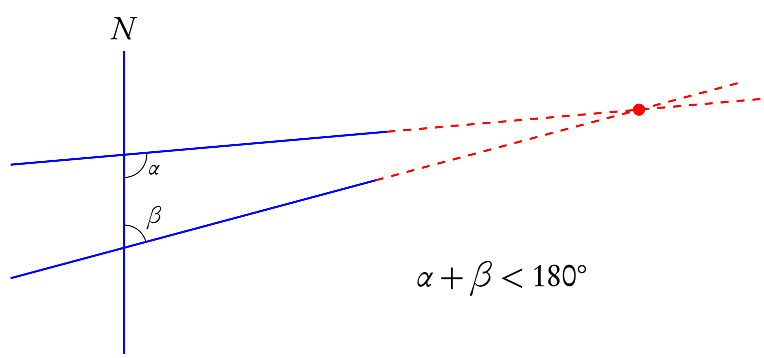
Philosophy of Mathematics

1. Euclid

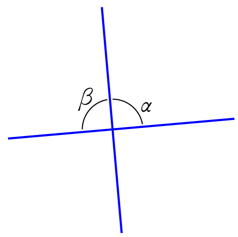
Around 300BC, Euclid wrote his *Elements*, which introduced the axiomatic method: showing that mathematical facts can be derived as *theorems* from a small number of unproven but obviously-true *axioms* (or postulates).

2. Euclid's axioms

1. Any two points can be joined by exactly one line segment
2. Any line segment can be extended to exactly one line
3. Given any point and any length, there is a circle whose radius is that length and whose center is that point.
4. Any two right angles are congruent
5. ("Parallel postulate") If a line N intersects two lines, and if the interior angles on one side of N add up to less than two right angles, then the lines intersect on that side of N



Once a theorem is proved, it can be used in proofs of other theorems. Also definitions can be used, such as "a right angle is any one of a pair of congruent adjacent angles formed from the intersection of two lines:

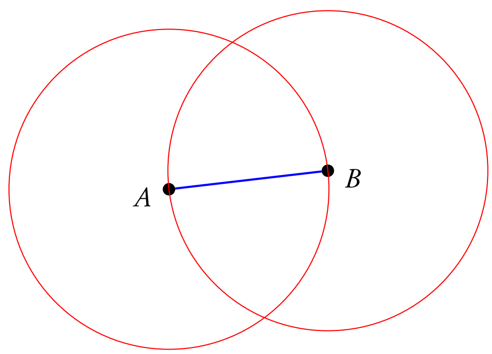


If $\alpha = \beta$, α and β are
defined to be right angles

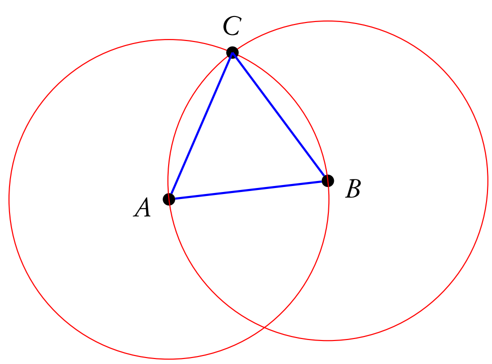
Examples of proving things from Euclid's axioms:

Theorem (Euclid's Proposition 1)**.** *For any two points, there is an equilateral triangle containing those points as vertices.*

Proof. Let A and B be any two points. By axiom 1, segment AB exists. By axiom 3, two circles with radius AB and centers A and B exist:



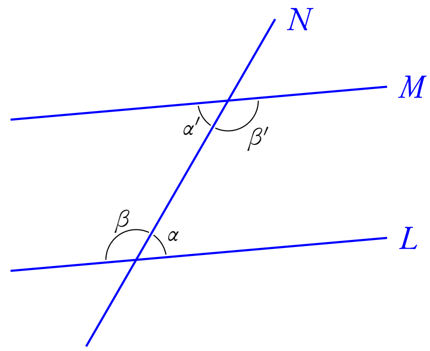
Where C is a point of intersection of the circles, by Axiom 1 segments AC and BC exist:



By definition of 'circle', $AB = AC$ and $AB = BC$. So $\triangle ABC$ is an equilateral triangle. □

Theorem (Euclid's Proposition 29). *If two parallel lines are cut by a transversal, alternate interior angles are congruent*

Proof.



The following argument shows that $\alpha = \alpha'$ and that $\beta = \beta'$.

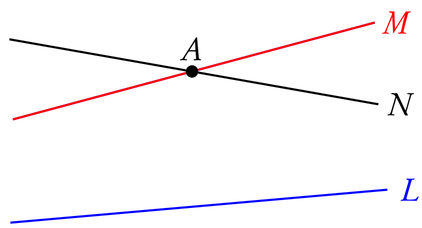
1. $\alpha + \beta' \geq 180°$ (otherwise by Axiom 5, M and L would meet, and hence wouldn't be parallel)
2. Similarly, $\alpha' + \beta \geq 180°$
3. But $\alpha + \beta = 180°$ (see below)
4. Similarly, $\alpha' + \beta' = 180°$ (again, see below)
5. From 1 and 3, $\alpha + \beta' \geq \alpha + \beta$, and so $\beta' \geq \beta$
6. From 1 and 4, $\alpha + \beta' \geq \alpha' + \beta'$, and so $\alpha \geq \alpha'$
7. From 2 and 3, $\alpha' + \beta \geq \alpha + \beta$, and so $\alpha' \geq \alpha$
8. From 2 and 4, $\alpha' + \beta \geq \alpha' + \beta'$, and so $\beta \geq \beta'$
9. From 5 and 8, $\beta = \beta'$
10. From 6 and 7, $\alpha = \alpha'$

□

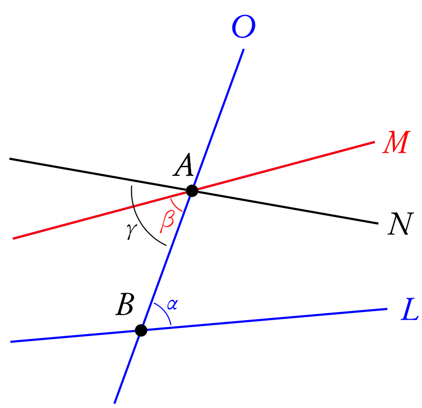
(This argument assumes that adjacent angles formed by the intersection of two lines add up to 180°, which would first need to be proved.)

Theorem (Playfair's axiom)**.** *Through any point not on a given line, there is at most one line parallel to the given line*

Proof. Suppose for reductio that there are two different parallels, M and N, to L through point A:



Choose some point, B, on L. By axiom 1, segment AB exists, which by axiom 2 can be extended to a line, O:



Since $M \neq N$ β is part of γ, and thus $\underline{\beta < \gamma}$.

But L and M are parallel lines cut by the transversal O; so by the previous theorem, $\alpha = \beta$. Similarly, since L and N are parallel, $\alpha = \gamma$. Thus $\underline{\beta = \gamma}$.

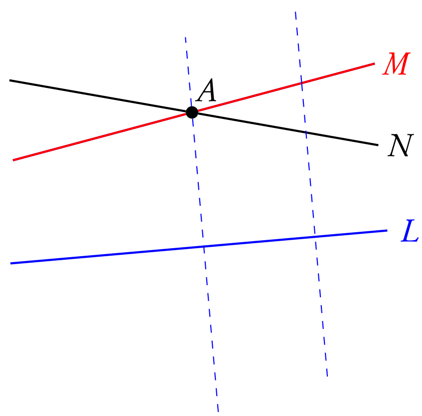
Contradiction. □

Playfair's axiom wasn't formulated or proven by Euclid. It is in fact equivalent to the parallel postulate.

3. Controversy about the parallel postulate

Axiom 5 seemed less obvious than the other axioms. Many tried to show it wasn't needed as an axiom by proving it from the other axioms. They all failed.

Example attempt: "there couldn't be two different parallels to L since they wouldn't both always remain the same distance from L"



Problem: this assumes without proof that "parallel lines are always the same distance apart", which turns out to be equivalent to the parallel postulate itself!

Other attempts similarly assumed things without proof which turned out to be equivalent to the parallel postulate, such as:

There exists at least one rectangle

There exists a pair of similar but not congruent triangles

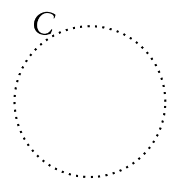
There is no upper limit to the area of a triangle

4. Independence: Poincaré disk model

In the late nineteenth century it was shown that the parallel postulate *cannot* be proven from the other axioms.

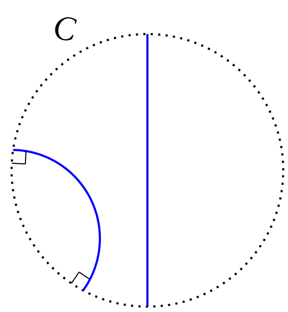
How do you show that a statement can't be proved? By finding a *model* in which the axioms are all true and the statement is false: a mathematical object that represents nonEuclidean space.

The model's parts will be parts of the interior of some circle, C:



The *points of the model* (i.e., the parts of the model that represent nonEuclidean points) are the points in the interior of C.

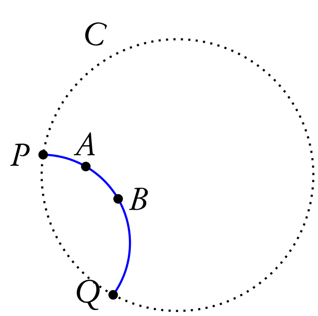
The *lines in the model* are i) the diameters of C (minus their endpoints), and ii) the arcs within C (minus their endpoints) of any circles that lie in the plane and intersect C at right angles:



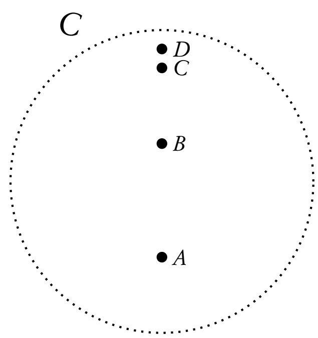
The *distance in the model* between points A and B is:

$$d(A,B) = \ln \frac{AQ \cdot BP}{AP \cdot BQ} \qquad \text{(“ln” = natural logarithm)}$$

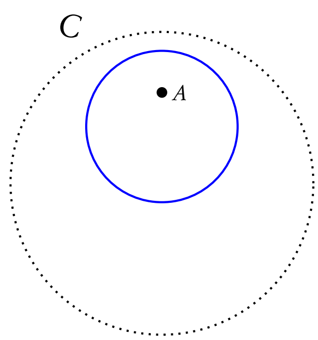
where P and Q are the points at which the line in the model containing A and B intersects C, and "AQ", "BP", etc., are "real" (Euclidean) distances.:



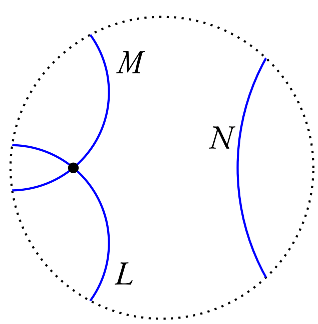
Note: distances become larger as you near the edge of the disk. For instance, A and B might be exactly as far apart as C and D:



Another illustration: a circle centered on A:



Main point: it can be shown that in the disk model, the first four axioms are true but the parallel postulate is false. Here is why the parallel postulate is false:



Why does this model, which *misinterprets* the words 'point', 'line', and 'distance', show that the parallel postulate can't be proved?

To say that a statement *S follows* from other statements $T_1, \ldots$—i.e., *logically* follows—is to say that:

> it is purely by virtue of *form* that whenever $T_1, \ldots$ are true, A is true.

If the parallel postulate followed from the other axioms, it would need to be true that no matter what 'point', 'line', and 'distance' meant, if the other axioms were true under those meanings, the parallel postulate would also be true under those meanings.

5. NonEuclidean geometries

Also in the nineteenth century, mathematicians began to investigate what would happen if you replaced the parallel postulate with other, incompatible postulates (perhaps making other adjustments to the Euclidean axioms, perhaps not).

Various of these "nonEuclidean" geometries were later shown to be consistent—using models, as before.

For instance, consider a model in which the points are the points on the surface of a sphere, and the lines are the great circles of the sphere. In this model, there are *no* parallels to a line through an external point.

6. NonEuclidean geometry and the a prioricity of geometry

Once these nonEuclidean geometries started becoming known, it became increasingly hard to believe that the geometry of physical space is a priori. How do we know, before doing observations, that physical space is structured as Euclid's geometry says, as opposed to being structured as one of the consistent nonEuclidean geometries says?

This became especially clear when Einstein's theory of general relativity said that the structure of physical space is in fact not Euclidean, but rather "curved" in a certain way. as neither a priori wrong nor a priori right; we needed to do science to figure out whether he was right. The geometry of physical space just isn't a priori.