Rationalist Realism

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Philosophy of Mathematics

1. Realism vs antirealism

Realism entities and facts are "out there" and independent of us.

One can be realist about one subject matter (e.g., physics) and anti-realist about another (e.g., etiquette).

Realist: Platonism, Fregean logicism. Anti-realist: Kant, Mill, formalism, intuitionism.

2. "Face-value" construal of mathematics

Mathematical statements have the meanings they appear to have—they're about mathematical entities.

3. Which objects?

"Reducing" mathematical objects to sets:

$$
\begin{aligned}
\text{imaginary numbers} &\Rightarrow \text{ordered pairs of real numbers}\\
\text{real numbers} &\Rightarrow \text{sets of rational numbers}\\
\text{rational numbers} &\Rightarrow \text{sets of ordered pairs of natural numbers}\\
\text{natural numbers} &\Rightarrow \text{sets (e.g., } \varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \ldots)\\
\text{functions} &\Rightarrow \text{sets of ordered pairs}\\
\text{ordered pairs} &\Rightarrow \text{sets (e.g., } \langle x, y\rangle = \{\{x,y\},\{x\}\})
\end{aligned}
$$

Set-theoretic definition of a group A group is an ordered pair $\langle A, *\rangle$ such that:

1. A is a nonempty set
2. $*$ is a two-place function, i.e., a set of ordered pairs $\langle\langle x,y\rangle,z\rangle$ such that for no x, y are there distinct elements $\langle\langle x,y\rangle,z\rangle$ and $\langle\langle x,y\rangle,z'\rangle$ in $*$. We abbreviate "$\langle\langle x,y\rangle,z\rangle \in *$" thus: "$z = x * y$".
3. For any $x, y \in A$, $x * y$ exists and is in A (closure)
4. For any $x, y, z \in A$, $(x * y) * z = x * (y * z)$ (associativity)
5. There exists an $e \in A$ such that for any $x \in A$, $x * e = e * x = x$ (Identity element)
6. For any $x \in A$ there exists some $y \in A$ such that $x * y = y * x = e$ (Inverses)

Since sets can in this way provide a foundation for all the rest of modern mathematics, what contemporary realists tend to be realists about is sets.

4. Axiomatic set theory

How to respond to Russell's paradox?

Naïve comprehension for any "condition", there exists a corresponding set—a set of all and only those things that satisfies the condition

This principle implied the existence of Russell's set::

$$R = \{x \mid x \notin x\}$$

So it must go. What "non-naïve" set theory should take its place?

The main answer has been "Zermelo-Frankel set theory". In place of Naïve comprehension it has "expansion" principles, which assert the existence of certain kinds of sets:

Null set There exists a set $\varnothing$ containing no members

Axiom of infinity There exists a set, A, that i) contains the null set, and i) is such that for any $a \in A$, $a \cup \{a\}$ is also a member of A. (Any such set A must be infinite, since it contains all of these sets: $\varnothing, \{\varnothing\}, \{\{\varnothing\}, \varnothing\}, \ldots.$)

Power set axiom For any set, A, the power set of A (i.e., the set of A's subsets) also exists

And it also has "contraction" principles, which take a set as given, and then say that we can use any chosen condition to pick out some corresponding set, of the same or smaller size as the given set. Example:

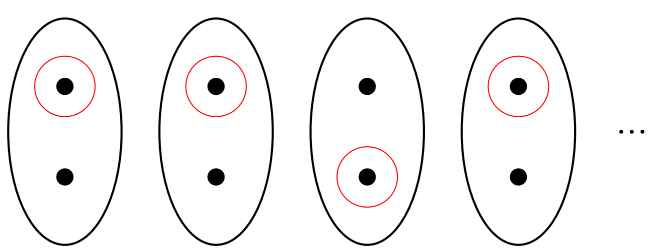
Axiom of separation Suppose some set A exists, and let C be any condition. Then there exists a set B consisting of all and only the members of A that satisfy the condition.

5. Nonconstructive proofs and objects

Realists accept nonconstructive proofs. Also they accept "nonconstructive entities".

Axiom of choice If A is a set of nonoverlapping sets, there exists some set C_A containing exactly one member of each member of A

A:



(C_A is the set of circled things)

We don't need the axiom of choice if we can pick out C_A by some explicit construction. E.g., if A is a set of sets of natural numbers, the set C_A can be defined this way:

$$C_A = \{x \mid \text{for some } a \in A,\ x \text{ is the least member of } a\}$$

We need the axiom of choice if we can't pick out C_A. Realists are happy with sets like these C_As.

6. Impredicative definitions

Russell's definition of the Russell set was:

$$R = \{x \mid x \notin x\}$$

We thus define R by reference to a plurality of objects including the very object R. Some say that such "impredicative" definitions are illegitimate.

But what about these impredicative definitions?

> Tall-P is defined as the tallest US president
>
> The least upper bound of a set, A, of real numbers is defined as the real number, y, such that i) y is an upper bound of A in the sense that $a \leq y$ for every $a \in A$, and ii) y is the *least* such upper bound in that $y \leq y'$ for any upper bound y' of A

Gödel distinguished three different sorts of impredicativity:

Self-involvement An object, x, contains itself

Self-presupposition A definition uses the very term being defined as part of the definition

Self-definition A definition of a term, T, quantifies over a range of objects that includes the object that T stands for

The third sort is unproblematic—for a realist anyway. For a realist, the objects in question all exist, and we're just selecting one of them to be denoted by T.

7. Gödel's epistemology

> [Russell] compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but their justification lies (exactly as in physics) in the fact that they make it possible for these 'sense perceptions' to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that (provided 'evidence' is understood in a sufficiently strict sense) this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. (Gödel, quoted in Shapiro p. 204)

Thus mathematical theories are justified because they "explain the data". But what is "the data"?

> Gödel wrote that principles of elementary arithmetic, such as basic equations and inequalities, have a kind of 'indisputable evidence that may most fittingly be compared with sense perception'. (Shapiro, p. 205)

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them … (Gödel, quoted in Shapiro p. 206)

Some have thought that this "mathematical intuition" is problematic—what is the physical mechanism?

8. Continuum hypothesis

Standard set theory is "ZFC": Zermelo-Frankel set theory plus the axiom of choice. There was a question about the following statement:

Continuum hypothesis ("CH") There is no set that is larger than the set of natural numbers but smaller than the set of real numbers

It wasn't clear whether CH was true or false.

But then Gödel proved in 1938 that the negation of CH isn't provable in ZFC; and Paul Cohen proved in 1963 that CH isn't provable either (assuming ZFC is consistent).

Deductivists can say: so what? But realists must admit that there is a question whose answer we don't know: whether CH is really true. Gödel accepted this, and said the following about how we might one day discover whether CH is true:

> a probable decision about [the] truth [of a proposed new axiom] is possible… in another way, namely, inductively by studying its 'success'. Success here means fruitfulness in consequences, in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs … A much higher degree of verification, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems … that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. (Shapiro, quoted on p. 210)