Set Theoretic Platonism: Epistemology and Application

Ted Sider
Philosophy of Mathematics

1. Epistemology: Gödel

How do we know (a priori) about sets? Gödel suggested that we *perceive* them:

> … despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them …(Gödel, 1983, 483–4)

But this is hard to believe, since perception is a causal process, and there are no causal connections between sets and our brains.

2. Epistemology: Quine

2.1 Holism in Science

Quine: empiricism, but knowledge can be based "indirectly" on the senses:

> Much of science involves universal generalizations ("all Fs are Gs"), which we don't learn from our senses directly.
>
> Mill's central epistemic principle was *enumerative induction*: we perceive many Fs, each of which is a G, and infer that (probably) all Fs are Gs.
>
> But enumerative induction can't account for the parts of science that concern unobservable entities, such as electromagnetic fields.
>
> Rather: we *hypothesize* unobservable entities; we develop a *theory* of how they behave; we then use our sense to check whether the theory's *predictions* are true.
>
> Finally, the *holism*: checking predictions confirms (or disconfirms) *entire theories*.

> The totality of our so-called knowledge or beliefs, from the most casual matters of geography and history to the profoundest laws of atomic physics or even of pure mathematics and logic, is a man-made fabric which impinges on experience only along the edges. Or, to change the figure, total science is like a field of force whose boundary conditions are experience. A conflict with experience at the periphery occasions readjustments in the interior of the field... But the total field is so undetermined by its boundary conditions, experience, that there is much latitude of choice as to what statements to re-evaluate in the light of any single contrary experience. No particular experiences are linked with any particular statements in the interior of the field, except indirectly through considerations of equilibrium affecting the field as a whole. (Quine, 1951, pp. 39–40)

"Periphery": beliefs directly connected to sensation. "Interior": beliefs less directly connected to sensation. When a theory has a good track record of making correct predictions, the entire theory is confirmed, even the interior, which may be about unobservable entities.

2.2 Holism applied to mathematical knowledge

Mathematical beliefs are in the interior. They take part in generating predictions. (Newton's laws are mathematical equations; only by using mathematics can we derive their prediction that planets move in elliptical orbits.) So they too can be indirectly confirmed by sensory information.

Challenge: what if some alternative body of beliefs contained no mathematics?

One such body contains *only* reports of observations. But this is "explanatorily bad" (like a body of belief saying that we are dreaming everything). It isn't obvious that there are mathematics-free explanatorily good alternative bodies of belief.

3. Applied mathematics

Challenge to Quine: how can mathematics be involved in making predictions if sets don't have physical properties?

Key to the answer: *impure sets*, i.e., sets containing nonmathematical objects as members, such as {Allen Iverson} and $\{75, \{\text{Mars}, \pi\}\}$.

Pure sets are built up solely from the null set. Part of the hierarchy of pure sets:

$$\begin{array}{cr} \varnothing & \text{(first stage)} \\ \varnothing \quad \{\varnothing\} & \text{(second stage)} \\ \varnothing \quad \{\varnothing\} \quad \{\{\varnothing\}\} \quad \{\varnothing, \{\varnothing\}\} & \text{(third stage)} \\ \text{etc.} & \end{array}$$

The *impure* hierarchy is similar, but at the first stage, in addition to the null set, there are all the sets whose members are nonsets.

Important example of impure mathematical entities: "impure functions", such as:

$$\begin{aligned} m(x) &= \text{ the mass of physical object } x \text{ in kilograms} \\ DP(x) &= \text{ the number of dollars in the pockets of person } x \end{aligned}$$

Given the definition of a function, these functions are just sets of ordered pairs.

$$m = \big\{\langle \text{Ted}, 77\rangle, \langle \text{Mars}, 6.39 \times 10^{23}\rangle, \dots\big\}$$

$$DP = \big\{\langle \text{Ted}, 5\rangle, \langle \text{Macklemore}, 20\rangle, \dots\big\}$$

And given the definition of an ordered pair, this can be rewritten:

$$m = \Big\{\big\{\{\text{Ted}\}, \{\text{Ted}, 77\}\big\}, \big\{\{\text{Mars}\}, \{\text{Mars}, 6.39 \times 10^{23}\}\big\}, \dots\Big\}$$

$$DP = \Big\{\big\{\{\text{Ted}\}, \{\text{Ted}, 5\}\big\}, \big\{\{\text{Macklemore}\}, \{\text{Macklemore}, 20\}\big\}, \dots\Big\}$$

Thus impure functions are simply impure sets.

We can use impure functions to state general laws, such as the ideal gas law:

For any ideal gas, x: $P(x) \cdot V(x) = n(x) \cdot R \cdot T(x)$

where P, V, T, and n are the following impure functions from physical objects to real numbers:

$$\begin{aligned} P(x) &= \text{ the pressure of } x \text{ in pascals} \\ V(x) &= \text{ the volume of } x \text{ in cubic meters} \\ T(x) &= \text{ the temperature of } x \text{ in degrees Kelvin} \\ n(x) &= \text{ the number of moles of } x \end{aligned}$$

and R is a certain real number (approximately 8.314).

Now we can show how mathematics can be applied even though mathematical objects don't stand in cause-effect relationships to physical objects. Suppose we learn by observation that for a certain ideal gas, g:

$$P(g) = 100000$$
$$V(g) = 17$$
$$n(g) = 700$$

And suppose we also know that the ideal gas law is true. We can infer from the ideal gas law, by logic, that:

$$P(g) \cdot V(g) = n(g) \cdot R \cdot T(g)$$

We can then infer (using our purely mathematical knowledge) that:

$$T(g) = \frac{P(g)V(g)}{n(g)R} = \frac{100000 \cdot 17}{700 \cdot 8.314} = 292$$

That is: the temperature of the gas g is 292° K (about 66° F).

How could we have learned by observation that, e.g., $V(g) = 17$? Quine's answer would be: this is indirectly supported by observations involving measuring instruments. The support is mediated by interior beliefs connecting impure functions to observable matters, such as:

> If one object, x, fits inside another object, y, then $V(x) \leq V(y)$.
>
> If a solid object, x, is not water soluble, and is submerged in an initially full container of water, and a quantity y of water spills out, then $V(x) = V(y)$.
>
> If some object z is made up of two nonoverlapping objects x and y, then $V(z) = V(x) + V(y)$.

4. Deductivism and applied mathematics, revisited

Deductivism says that a statement of pure mathematics, S, isn't meaningful; what is true is the statement that S can be proved from certain axioms. What will the deductivist's account of impure mathematical statements, such as 'm(Ted) = 77', be?

How about ‘ “$m(\text{Ted}) = 77$” is provable from the axioms of pure mathematics’? No: the axioms say nothing about the particular function m.

How about ‘ “$m(\text{Ted}) = 77$” is provable from the axioms of pure mathematics plus M’, where M are some new axioms about m? But then the new axioms wouldn’t be arbitrarily chosen; they must be the *right* ones.

5. Continuum hypothesis

Continuum hypothesis

There is no set that is larger than the set of natural numbers but smaller than the set of real numbers

In 1938, Gödel showed that it can’t be *dis*proved from the ZF axioms. In 1963, Paul Cohen showed that it can’t be proved either. But set-theoretic platonists seem to face the question of whether it is *true*.

Gödel suggested that we might eventually get evidence for or against it:

> There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems … that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. (Gödel, 1983, p. 477)

But that (arguably) hasn’t happened.

Some set-theoretic platonists simply live with the ignorance. Others defend “set theoretic pluralism”, according to whichthere are many “set-theoretic universes”, some of which obey the continuum hypothesis and some of which don’t.

References

Gödel, Kurt (1983). “What is Cantor’s Continuum Problem?” In Paul Benacerraf and Hilary Putnam (eds.), *Philosophy of Mathematics*, 2nd edition, 470–85. Cambridge: Cambridge University Press. Expanded version of a paper that originally appeared in *The American Mathematical Monthly* 54 (1947): pp. 515–25.

Quine, W. V. (1951). "Two Dogmas of Empiricism." *Philosophical Review* 60: 20–43. Reprinted in Quine 1953: 20–46.

— (1953). *From a Logical Point of View*. Cambridge, MA: Harvard University Press.