Hamkins on Set-theoretic Pluralism

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1. The multiverse view

Multiverse view "There are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe." (Hamkins, 2012, p. 416)

Compare:

Full-blooded Platonism "…any mathematical object which possibly *could* exist actually *does* exist." (Balaguer, 1995, p. 303)

Principles of plenitude for material objects (E.g., unrestricted composition, and modal variants.)

They allow for some nonobjectivity while securing an underlying objectivity.

2. Independence proofs

Continuum hypothesis There is no set intermediate in size between the set of natural numbers and the set of real numbers.

Neither ∼CH (Gödel, 1940) nor CH (Cohen, 1963) can be proved from the ZF axioms.

One way to prove that a claim can't be proved from the ZF axioms is to construct a set-theoretic model in which the axioms are true but the claim is false.

Another way: show that the axioms are true and the claim is false when the quantifiers are restricted to sets of a certain sort.

Let Tr(ϕ) be the result of restricting all quantifiers in ϕ with the predicate Cx, for "constructibility" (a notion defined by Gödel). E.g.:

$$\text{Ext}: \quad \forall x \forall y \big(\forall z(z \in x \leftrightarrow z \in y) \to x = y\big)$$

$$\text{Tr(Ext)}: \quad \forall x \forall y \Big(\big(Cx \& Cy\big) \to \Big(\forall z\big(Cz \to (z \in x \leftrightarrow z \in y)\big) \to x = y\Big)\Big)$$

"For any constructible sets, x and y: if x and y have exactly the same constructible sets as members, then $x = y$."

Gödel showed that the translations of the ZF axioms, and also CH, are all theorems of ZF. This shows that $\sim$CH isn't provable from the ZF axioms:

> Suppose there is a proof P of $\sim$CH from the axioms of ZF.
>
> Now change each formula, ϕ, in P to its translation $\mathrm{Tr}(\phi)$.
>
> The resulting formulas constitute (near enough) a legal proof of Tr($\sim$CH) from the translations of the ZF axioms. (This is because the process of translation preserves the logical validity of inferences.)
>
> So since the translations of the ZF axioms are theorems of ZF, Tr($\sim$CH), is a theorem of ZF.
>
> But Tr(CH) is a theorem of ZF. Since Tr($\sim$CH) is $\sim$Tr(CH), ZF is therefore inconsistent, given the initial supposition.
>
> Conclusion: if ZF is consistent, there is no proof of $\sim$CH from the axioms of ZF.

Gödel's "constructible universe" (the collection of constructible sets) is an "inner model", whose "domain" contains only some of the sets. Cohen's proof that CH can't be proved from the ZF axioms uses an "outer model", whose domain, roughly speaking, contains new sets beyond the "real ones". More exactly: he uses the axioms of set theory to construct sets of sentences that *say* that there exist new sets, and proves that those sets of sentences are consistent.

3. The argument for the multiverse view

> At first, [nonEuclidean] geometries were presented merely as simulations within Euclidean geometry, as a kind of playful or temporary reinterpretation of the basic geometric concepts… useful perhaps for independence results, for with them one can prove that the parallel postulate is not provable from the other axioms. In time, however, geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them, and gradually they accepted the alternatives as geometrically meaningful. Today, geometers have a deep understanding of the alternative geometries, which are regarded as fully real and geometrical.
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> The situation with set theory is the same. The initial concept of set put forth by Cantor and developed in the early days of set theory seemed

> to be about a unique concept of set, with set-theoretic arguments and constructions seeming to take place in a unique background set-theoretic universe. Beginning with Gödel's constructible universe L and particularly with the rise of forcing, however, alternative set-theoretic universes became known, and today set theory is saturated with them. Like the initial reactions to non-Euclidean geometry, the universe view regards these alternative universes as not fully real, while granting their usefulness for proving independence results. Meanwhile, set theorists continued, like the geometers a century ago, to gain experience living in the alternative set-theoretic worlds, and the multiverse view now makes the same step in set theory that geometers ultimately made long ago, namely, to accept the alternative worlds as fully real.
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> A stubborn geometer might insist—like an exotic-travelogue writer who never actually ventures west of seventh avenue—that only Euclidean geometry is real and that all the various non-Euclidean geometries are merely curious simulations within it… Similarly, a set theorist with the universe view can insist on an absolute background universe V, regarding all forcing extensions and other models as curious complex simulations within it… Such a perspective may be entirely self-consistent, and I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real. (Hamkins, 2012, pp. 425–6)

Some questions:

1. How much are we relying on mathematical experience?
2. What is the relevance of "missing out on the insights"?
3. What do "fully geometrical", "fully real", etc., mean?

4. Stating the multiverse view

There are questions about how to state the view. E.g., if Balaguer's "any mathematical entity that can exist, does exist" just means "if a mathematical theory is consistent, then it has a model", then it's trivially true (for first-order theories, anyway).

It would be nice to have a "once-and-for-all" statement of what mathematical reality is like, from which one can construct all talk of the multiverse (compare Sider (2020, Chapter 5) on "quotienting"). That seems unavailable.

5. Motivation for the multiverse view

What is the point of accepting the multiverse view, rather than something like Balaguer's view?

References

Balaguer, Mark (1995). "A Platonist Epistemology." *Synthese* 103: 303–25.

Hamkins, Joel David (2012). "The Set-Theoretic Multiverse." *Review of Symbolic Logic* 5(3): 416–449.

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