A Foundational Crisis

Ted Sider
Philosophy of Mathematics

1. Foundations

There was a crisis in the foundations of mathematics around the beginning of the 20th century, centered on the notion of infinity.

2. Actual versus potential infinity (Aristotle)

Actual infinity an infinite collection of objects, all of which actually exist.

Potential infinity a collection of objects that always *can* be increased further

Some confusing things about infinity:

Hilbert's Hotel: (a full infinite hotel frees up a room by having each occupant move up one floor)

Zeno's paradox (one of them): motion is impossible, because in order to reach a point one meter away, you'd need to do infinitely many things: travel $\frac{1}{2}$ meter, then travel $\frac{1}{4}$ meter, then travel $\frac{1}{8}$ meter, and so on.

(Aristotle's solution: you don't need to do all of these things, since the one-meter portion of space is merely potentially infinitely divisible.)

Historically, mathematicians didn't think too seriously about actual infinities, but infinite collections started showing up in the foundations of mathematics in the nineteenth century. Here are some of the places where they showed up.

3. Real numbers and infinity

The *rational* number line has "gaps". It contains points that get closer and closer to $\sqrt{2}$, namely: $1.4, 1.41, 1.414, \dots$, but not $\sqrt{2}$ itself (which is the *limit* of this series).

The defining feature of the *real* line (in its modern conception), ensures that there are no such gaps:

Completeness: if a set of numbers has an upper bound, it has a *least* upper bound

(A “bound” of S is a number that is $\geq$ every member of S.)

The sets S that are relevant here are infinite sets.

4. Functions

A *function* is a “rule” which yields a unique number as “output”, when given any number as “input”.

Functions were long regarded as merely being formulas. For example, the squaring function is just the formula“$f(x) = x^2$”.

But on a more modern conception, a function is any “arbitrary correlation” between inputs and outputs.

Arbitrary correlations were eventually defined as infinite lists of input-output pairs (each input is paired with its corresponding output).

5. Nonconstructive proofs

Most traditional mathematical proofs are “constructive”: to show that a certain kind of object exists, you construct some particular object of that sort. Example of a *non*constructive proof:

Proof that some nonzero digit occurs infinitely often in the decimal expansion of π. Suppose for reductio that each of the digits 1–9 occurs only finitely many times. Then once all those digits are done occurring, from that point onward the decimal expansion would consist only of 0s. π would therefore be a rational number. Since π is known to be irrational, we have a contradiction. □

This is nonconstructive because it doesn’t say (or give a method for figuring out) *which* digit occurs infinitely often.

The connection to infinity is this: in a nonconstructive proof, we consider some actually infinite set, and argue that given its general features, it must contain at least one object of the specified sort.

6. Cantor on sizes of infinity

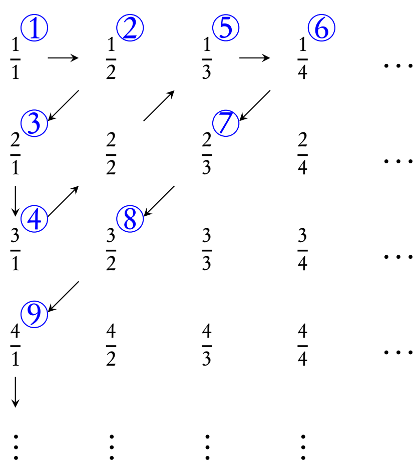
Mathematicians originally thought of sets as trivial/obvious. Then Cantor discovered that some infinite sets are bigger than others.

Examples of infinite sets that are the *same* size (equinumerous):

- Even natural numbers $\approx$ natural numbers:

$$\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ 2 & \longleftrightarrow & 4 \\ 3 & \longleftrightarrow & 6 \\ 4 & \longleftrightarrow & 8 \\ \vdots & & \vdots \end{array}$$

- Rational numbers $\approx$ natural numbers:



$$\begin{array}{ccc} 1 & \longleftrightarrow & \frac{1}{1} \\ 2 & \longleftrightarrow & \frac{1}{2} \\ 3 & \longleftrightarrow & \frac{2}{1} \\ 4 & \longleftrightarrow & \frac{3}{1} \\ 5 & \longleftrightarrow & \frac{1}{3} \\ 6 & \longleftrightarrow & \frac{1}{4} \\ 7 & \longleftrightarrow & \frac{2}{3} \\ 8 & \longleftrightarrow & \frac{3}{2} \\ 9 & \longleftrightarrow & \frac{4}{1} \\ \vdots & & \vdots \end{array}$$

But Cantor proved that there are more *real* than natural numbers. Here is his proof that the set of real numbers between 0 and 1 $\not\approx$ natural numbers. Suppose for reductio that there is some one-to-one correspondence between these sets:

$$\begin{array}{ccc} 1 & \longleftrightarrow & r_1 \\ 2 & \longleftrightarrow & r_2 \\ 3 & \longleftrightarrow & r_3 \\ \vdots & & \vdots \end{array}$$

Since each real number can be represented as an infinite decimal, we can represent the one-to-one correspondence thus:

$$\begin{array}{cccccccc} 1 & \longleftrightarrow & 0. & a_{11} & a_{12} & a_{13} & \ldots \\ 2 & \longleftrightarrow & 0. & a_{21} & a_{22} & a_{23} & \ldots \\ 3 & \longleftrightarrow & 0. & a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \end{array}$$

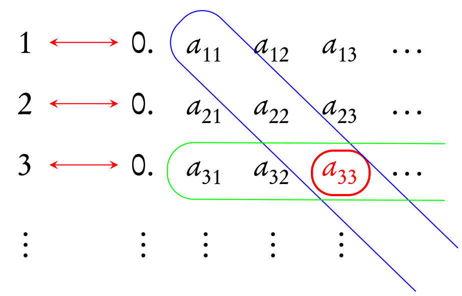
where each a_{ij} is a digit. Now construct a new infinite series of digits, which at each position differs from the "diagonal sequence":

$$\begin{array}{cccccccc} 1 & \longleftrightarrow & 0. & \boxed{a_{11}} & a_{12} & a_{13} & \ldots \\ 2 & \longleftrightarrow & 0. & a_{21} & \boxed{a_{22}} & a_{23} & \ldots \\ 3 & \longleftrightarrow & 0. & a_{31} & a_{32} & \boxed{a_{33}} & \ldots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \end{array}$$

The digits d_i of this new series are::

$$d_i = \begin{cases} 7 \text{ if } a_{ii} = 6 \\ 6 \text{ if } a_{ii} \neq 6 \end{cases}$$

Now consider the real number $d = 0.d_1 d_2 d_3 \ldots$. Since it is between 0 and 1, and since we have supposed that the one-to correspondence correlates each real number with some natural number, d must be equal to some r_i. But consider the digits of r_i and d at the i^{th} spot—the spot where the decimal expansion of r_i intersects the diagonal sequence:



The i^{th} digit of r_i is a_{ii}, whereas the i^{th} digit of d_i is 7 if a_{ii} is 6 and 6 if a_{ii} is not 6. Thus $d \neq r_i$. Contradiction.

Cantor also showed that for any set A, the *power set* of A—that is, the set of all A's subsets, is larger than A.

7. Set-theoretic paradoxes

Later, *paradoxes* were discovered in "set theory": seemingly correct arguments leading to contradictions. The simplest was Bertrand Russell's. Let R be the set containing all and only the sets that *aren't members of themselves*. Thus:

i) Any set that is *not* a member of itself *is* a member of R

ii) Any set that *is* a member of itself is *not* a member of R

But now, either R a member of itself or it isn't. But:

> If R *is* a member of itself, then by ii) it wouldn't be a member of R, and so wouldn't be a member of itself, which is a contradiction.
>
> And if R *isn't* a member of itself, then by i), R would be a member of R, and so it would be a member of itself after all—another contradiction.

Eventually, a theory of sets was found that avoided the paradoxes. But for a long time, the field was in crisis.