Formalism and Deductivism

Ted Sider
Phil of Math

Formalism

"Mathematics is about nothing more than the rule-governed manipulation of symbols"

- A possible reaction to the paradoxes
- Seems to dissolve puzzles about the epistemology and metaphysics of mathematics
- Suggested by how we often just memorize techniques for calculation
- Suggested by the abstract approach to mathematics and modern logic

1. Game formalism

Game formalism

Mathematical language is meaningless; mathematics consists of making moves in a game about nothing.

Comparison with chess. Mathematics is a game played on paper, whose pieces are symbols like '0', '1', 'm', 'n', '$+$', and '$\times$'.

1.1 The problem of arbitrariness

If mathematics is just a game, its rules are arbitrary.

Although the choice of axioms in abstract mathematics *is* arbitrary, the use of logic in mathematical proofs isn't.

1.2 The problem of application

If mathematical language is meaningless, how can it be applied?

("Game theory" studies games which needn't have meaning, and can be applied to the real world; but that mathematics itself is meaningful.)

2. Term formalism

Term formalism

Mathematical objects are just terms (symbols). Mathematical terms denote themselves.

For example, the number `0` is just the numeral '`0`'. '`0`' denotes '`0`'.

- Term formalism does *not* say that mathematics is meaningless
- Nevertheless it still seems to solve the epistemological and metaphysical puzzles about mathematics.

2.1 Reinterpreting '='

Problem: term formalism seems to imply that this sentence is false:

$$2+2=3+1$$

since the left and right hand sides denote different terms ('$2+2$' $\neq$ '$1+3$').

Solution: reinterpret "$a=b$" to mean that a and b are "equivalent".

2.2 The theory of terms is mathematical

Term formalists (in the case of arithmetic) need:

- A rigorous definition of which terms are to count as natural numbers
- A rigorous definition of 'equivalent'
- A rigorous theory of how terms in general—"strings"—behave. Given this theory and the definitions above, it must be possible to prove the axioms of arithmetic.

But now the terms are very number-like (for instance, there are infinitely many of them). How do we know about them a priori? The old epistemological problems have returned.

(This point is driven home by the fact that the "terms" need to be term *types*.)

2.3 Real numbers

What kinds of terms will the term formalist identify with real numbers?

Problem: many (indeed, most) real numbers are "transcendental" (not roots of polynomials with integral coeffecients), and aren't denoted by any term in our language.

Arguably, *no* language could contain a term for every language:

Fact 1 (Cantor): there are more real numbers than natural numbers

Fact 2 (see below): in any language, there are exactly as many terms as natural numbers

Let L be any language.

Assumptions: L has finitely many primitive symbols and rules; all terms of L are finitely long.

We can then construct a one-to-one correspondence between the terms of L and natural numbers by making a numbered list of all the terms of L. Do it in stages:

Stage 1: add all the terms of length one

Stage 2: add all the terms of length two

Stage 3: add all the terms of length three

etc.

3. Deductivism

Deductivism

Mathematics investigates the logical consequences of axioms whose nonlogical expressions are uninterpreted

Thus mathematics establishes conditional statements:

> "if <axioms>, then <theorem>"

which would be true no matter how we interpreted the nonlogical expressions.

Advantages/features:

> No need to explain how we know the axioms
>
> No need to explain what mathematics is about
>
> Doesn't imply that proofs are arbitrary
>
> Meshes with/relies on abstract/formalized conception of mathematics

4. Deductivism and applied mathematics

If we can give a scientific interpretation to mathematical language under which the axioms are true, we would then know that the theorems are true.

This might work for geometry.

But suppose we want to use arithmetic for counting place-settings:

> How to interpret arithmetic predicates?
>
> How will the axioms come out true, given that there are only finitely many place-settings?

5. Deductivism and mathematical logic

Is knowledge of what follows from axioms really unproblematic?

The problem is sharpened if we "mathematize" logic, as Hilbert did—if we give mathematically rigorous definitions of logical concepts. For example:

Definition of proof

A *proof of conclusion* C *from premises* $P_1, \ldots, P_n$ is defined as a finite sequence of formulas, the last of which is C, in which each formula is either i) an axiom of logic, or ii) one of the premises $P_1, \ldots, P_n$, or iii) follows by a rule of inference from earlier formulas in the series.

We can similarly give mathematical definitions of *formula*, *axiom of logic*, and *rule of inference*. They're all a matter of the "shapes" of strings of symbols.

Then we define "following from":

Definition of following from

A formula C *follows from* formulas $P_1, \ldots, P_n$ if and only if there exists some proof of C from $P_1, \ldots, P_n$

Thus deductivists are committed to the *truth* of statements like these:

> There exists some proof—that is, a certain sequence of strings—of the string '$\forall x \forall y\, x + y = y + x$' from the strings that are the axioms of arithmetic

So at least one portion of mathematics can't be the mere investigation of the logical consequences of uninterpreted axioms, namely, the theory of strings.