Foundations of Mathematics

Philosophy 423, Fall 2020
Online, T/Th 1:10–2:30
office hours Th 2:30–3:30
& by appt

A study of the set-theoretic approach to the foundations of mathematics, and connections to the philosophy of mathematics and logic.

Sets are collections. The basic idea is simple, but it requires careful mathematical development, in light of Russell's and other paradoxes. The idea is powerful, since mathematical objects can be modeled as sets. For these reasons sets have been central in the foundations of mathematics. Also set theory is mathematically important in its own right, for instance in its account of infinity.

In set theory proper we will cover basic concepts of set theory, the axiomatic approach, the construction of natural numbers, integers, rational numbers, and real numbers within set theory, ordinals and cardinals, transfinite induction and recursion, and models of set theory within set theory. Philosophically we will discuss the axiomatic approach in general, questions about foundations of mathematics, questions about the nature of sets, set theoretic paradoxes (Russell's, Skölem's), the ontology of set theory, the significance of alternative options for construction, the recursion theorem and recursive definitions in logic, and the philosophical significance of models of set theory within set theory.

Prerequisite

01:730:315 or 01:730:407 or 01:640:300 or by instructor permission. Note: this course is difficult (much, much harder than—and very different from—intro logic). You need to be comfortable with abstract mathematical reasoning (of the sort involved in advanced logic classes or mathematics classes like linear algebra or group theory). Please see me if you're unsure whether this class is appropriate for you.

Readings

Textbook: Herbert Enderton, *Elements of Set Theory*, ISBN-13: 978-0122384400. My class notes are fairly complete, are in some cases more explicit than the textbook, and will be available on Canvas. However, the homework problems are in the textbook, not my notes.

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Requirements

Two exams (70%), plus periodic homework assignments (30%). The first exam will be on **October 27**. The second exam, which will cover only the second half of the course, will be during finals week, date and time TBA. Homework assignments will be posted on the Canvas site:

`https://rutgers.instructure.com/courses/67753`

You must do your homework completely on your own: no working in groups, no consulting outside sources (e.g., other books, the internet). Some homework problems will be difficult; I don't expect everyone to be able to solve them all. Just do the best you can. If you get stuck on a problem, you can ask me for a hint. Homework must be produced electronically (not hand-written) and turned in at the Canvas site. You are also responsible for insuring that your homework is successfully uploaded. **Late homework will be penalized 10%; after 3 days it will not be accepted.**

Learning goals

Students will learn basic axiomatic set theory, the philosophical issues it raises, connections to metalogic, and the use of set theory in the foundations of mathematics. Students will also develop skills in constructing and presenting abstract mathematical proofs.

Schedule

Week 1 Intro to sets; Russell's paradox; the axiomatic method and the iterative hierarchy; first axioms; axioms and the paradoxes; establishing that sets exist; generalized unions and intersections. (Enderton chapters 1–2, but skim pp. 27–32; notes parts I and II.)

Week 2 Relations and functions; axiom of choice; equivalence relations. (Begin chapter 3, but skip "infinite cartesian products", pp. 54–5; notes sections 7–10.)

Week 3 Quotient sets; orderings; philosophical background on constructing mathematical objects; construction of natural numbers; axiom of infinity. (Finish chapter 3; start chapter 4; notes sections 11, 12.)

Week 4 Transitive sets; recursion theorem. (Continue chapter 4, skipping material on Peano systems, though note that the material on transitive sets is in the section entitled "Peano's Postulates"; notes section 13.)

Week 5 Addition, multiplication, ordering for natural numbers. (Finish chapter 4; notes section 14.)

Week 6–Week 7 Construction of integers, rational numbers, and real numbers (Chapter 5; notes part V.)

Week 8 Equinumerosity and finitude. (Begin chapter 6; notes sections 18–19)

Week 9 Cardinal arithmetic; cardinal ordering; S/B theorem. (Continue chapter 6; notes sections 20–22.)

Week 10 Equivalent statements of the axiom of choice; continuum hypothesis; independence proofs; some philosophical issues. (Finish chapter 6 but skip 159–61 and skim 162–5; notes sections 23–26.)

Week 11 Well-orderings; transfinite induction and recursion. (Begin chapter 7; notes sections 27–29.)

Week 12 Replacement axioms; proof of recursion theorems; ordinal numbers. (Continue chapter 7; notes sections 30–35.)

Week 13 More on ordinal numbers; well-ordering theorem; definition of cardinals. (Continue chapter 7; notes sections 36–38.)

Week 14 Rank; Regularity axiom; natural models. (Finish chapter 7; skim pp. 249–56; notes section 39 and part VIII.)