Things to keep in mind while doing set-theory proofs

1. Keep in mind the "canonical" methods for establishing statements of various logical forms. For instance, suppose you want to show that $A \subseteq B$. By definition, what you're trying to show is this: "for any x, if $x \in A$, then $x \in B$". And the way you establish such a statement is as follows. First you say "consider any object x that is a member of A" (or perhaps just "let $x \in A$"). Then you try to use this information to show that $x \in B$. When you succeed, you will have established what you were trying to show (since you chose an arbitrary member of A and showed that it would need to also be in B).

2. Another tip on the use of logic. First, suppose you know a statement of the form "P or Q". The way you *use* such a statement is to "separate cases". That is, suppose you're trying, ultimately, to prove some statement S. You would then reason as follows. "I know that P or Q. So first suppose P. For the following reasons, S would then be true…Suppose on the other hand Q. Then for the reasons, S would again be true…So either way, S is true."

3. Yet another logical tip: if you are using a variable (like 'x'), always be sure you're clear where that variable is coming from, and for how long its use is valid. For example, suppose you're trying to show that $A \subseteq B$. So you begin by saying "let $x \in A$", and then you try to show that $x \in B$. In this part of your proof it is ok to keep talking about x, since x is the arbitrarily chosen member of A that you're showing is a member of B. But once you succeed in showing that $x \in B$, and have concluded that A is indeed a subset of B, you shouldn't continue talking about x. The variable x was just being used to stand for an arbitrary member of A *within the context of a particular part of your proof*, and has no validity outside of that context.

(Well, let me qualify the prohibition on re-using the variable 'x' after completing the subproof in which it was introduced. It would be ok to re-use that same variable later on, to stand for some newly chosen arbitrary object perhaps. But it would not then be ok to re-use information about the "old x"—i.e., information from within the earlier subproof—in this new context.)

4. The most straightforward method for showing that two sets A and B are identical is the brute force method: first choose $x \in A$ and show that $x \in B$; and then choose $x \in B$ and show that $x \in A$. (That shows that A and B have exactly the same members, and so, by Extensionality, they are identical.) But sometimes a quicker method is to show that the condition for being a member of A is

equivalent to the condition for being a member of B. And a quick way to do *that* is sometimes the "round-robin" method for proving strings of biconditionals:

$$\begin{aligned} x \in A &\text{ iff } \phi_1(x) \\ &\text{ iff } \phi_2(x) \\ &\text{ iff } \phi_3(x) \\ &\text{ iff } x \in B \end{aligned}$$

You begin by writing down a biconditional statement: x is a member of A if and only if a certain condition, ϕ_1, holds of x. (Obviously you can write down this step only if it's justified—only if, given what you know about set A, something is a member of A iff it is ϕ_1.) Then, for each subsequent step, you write down a new statement to the right of "iff" that is equivalent to the previous right-hand side. For example, you'd be justified in writing down the second line if you knew that conditions ϕ_1 and ϕ_2 are equivalent—i.e., if you know that an object x satisfies ϕ_1 iff it satisfies ϕ_2. Eventually, you try to have the right-hand-side be that $x \in B$ (the last line will be justified if being a member of B is equivalent to satisfying ϕ_3). Equivalence is transitive; so the final right-hand side (i.e. $x \in B$) holds iff the original left hand side holds ($x \in A$). And so, by Extensionality, $A = B$.

Note: when using the round-robin method, you need to show that each of the conditions holds *if and only if* the preceding one does, and not merely that each holds *if* the preceding one holds. For instance, you don't merely need to show that if ϕ_1 is true, then ϕ_2 is true. You also need to show that ϕ_1 is true *only* if ϕ_2 is true—or, equivalently, that if ϕ_2 is true then ϕ_1 is true.