Crash course on quantities and measurement theory*

Ted Sider Fall 2011

1. The problem of quantity

We can divide properties into *qualities* and *quantities*.[1] (Similarly for relations.) Qualities don't come in degrees, quantities do. *Being a US citizen* is a quality. You either are or aren't a US citizen (as they say). Mass is a quantity: you can have more or less mass.

When metaphysicians think about properties, they tend to think about qualities. (Perhaps this is because our language for foundational work is predicate logic.) But in science, especially physics, the most important properties are quantities: mass, charge, distance, etc.

When scientists speak of quantities, they do so using numbers. Not only do they name particular quantities using names for numbers: "5 g mass". Their theories also make essential use of the numbers. The law $F = ma$ says that, given suitable choices of unit, a number representing the net force on a particle is equal to the result of multiplying the numbers representing its mass and acceleration.

Quantities raise an important metaphysical issue: what are the fundamental facts of quantity like, which enable them to be spoken of and theorized about using numbers? This is the kind of issue that metaphysics must address.

The question can be made pressing by considering the simplest theory of quantity: that quantities are relations to numbers. On this view, the fundamental property of mass is in fact a relation, not a property: a relation between concrete material objects and real numbers. The relation is, perhaps, the mass-in-kilograms relation, which holds between concrete object x and real number r iff x's mass is r kg.

This theory makes excellent and straightforward sense of the use of numbers to represent quantities in science. But it is metaphysically problematic, for two main reasons.

*A standard text on measurement theory is Krantz et al. (1971); for a useful introduction to the metaphysics of quantity, see Eddon (2013).

[1] 'Quality' here has nothing to do with the idea of "qualitative properties" meaning properties that don't involve any particular individuals.

First, this theory involves real numbers in the facts of mass. The fundamental mass facts involve real numbers no less than they involve concrete objects. Isn't that weird? Attempts to bring out the weirdness:

i) real numbers are abstract, so causally inert, so can't be involved in laws of nature.

ii) real numbers don't fundamentally exist, and so mass can't fundamentally make reference to them

iii) real numbers are constructed entities; and constructed entities can't be involved (qua the construction) in fundamental facts

Are these decisive objections? Suppose you thought that real numbers exist fundamentally and aren't constructed. Then ii) isn't a problem. And i) might be regarded as based on unjustified dogmatic beliefs about abstracta. Why can't abstracta be involved the laws? I'm inclined to think that iii) remains a problem, though (see my Sider (1996)).

The second main reason—to my mind the more serious one—to worry about the theory is that seems to privilege a single unit of mass. Suppose for the sake of argument that M assigns masses in kg. That is, objects that bear M to the real number 1 are 1 kg in mass. In that case, the kg scale seems to be objectively privileged in a way that other scales, such as the pounds scale, are not. Put another way: consider all the functions from massive objects to real numbers, differing from one another only in their scale. According to the simple theory of the nature of mass, one of these functions is objectively privileged, as being the fundamental mass relation.

Could this worry be answered? Could we say that, where M is a fundamental mass relation to numbers, each "scalar transformation" of M is also fundamental? In some cases, in order to avoid arbitrariness we ought to admit some "redundant" structure. Perhaps, for example, we ought to claim that both parthood and overlap (say) are fundamental relations. Or, perhaps we ought to say that in addition to negation, both conjunction and disjunction are fundamental concepts. For it would be arbitrary to say that one but not the other is fundamental. But in the case of the scalar transformations of M, we would be accepting infinitely many fundamental relations, each of which suffices to describe the facts (and in a way that's exactly parallel to each of the others).

So: we have seen that the simple theory, according to which quantities are relations to numbers, is metaphysically problematic. A main metaphysical puzzle about quantity, then, is this: why are numbers so useful in talking about quantities, if they're not involved in the fundamental mass facts in the way that the simple theory says?

There is another (related) metaphysical puzzle too. As we pointed out earlier, it's useful in science to measure quantities with real numbers: 5 kg, 7 mm, etc. But consider:

1. The mass of object *o* is 5
2. The mass of object *o* is 5 g
3. The mass of object *o* is greater than the mass of object *p*
4. The mass of object *o* is twice the mass of object *p*
5. The mass of object *o* is greater than the charge of object *p*
6. Smith is witty to degree 6.808942 in the Johnson scale
7. The wit of Smith is greater than the wit of Jones
8. The wit of Smith is twice the wit of Jones

The first statement on this list doesn't "make sense". Why? Because you need to specify a scale in order to use numbers to measure mass—there's no such thing as having mass 5 absolutely, so to speak. (2) does make sense because it specifies a scale. (3) also makes sense, even though it doesn't specify a scale, because whether one thing is more massive than another doesn't depend on the scale. Similarly for (4). (5), though, doesn't make sense: there are no absolute comparisons of mass with charge—depends on the scales chosen. (6) doesn't make sense either, but for a different reason: there just couldn't be a scale for wit that assigned such precise numbers. It's not that wit isn't a quantity at all—(7) does seem to make sense (at least in some cases). But note that (8) doesn't make sense: unlike for mass, it doesn't make sense to say (literally) that someone has twice as much wit as another.

The puzzle (or question) is: what's going on here? What does it mean to say that some of these uses of numbers to measure quantities don't "make sense"; and why do some kinds of uses of numbers (twice-as-much-as) make sense for some quantities (mass) and not others (wit)?

2. The idea of measurement theory

Measurement theory is a theory of the use of numbers to measure quantities. It was developed primarily by philosophers of science, who had epistemological concerns in mind; but it can also be used in metaphysics.

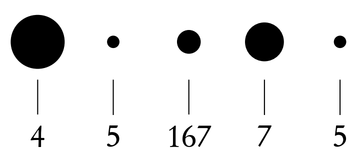
The basic idea is that numbers can be used to *represent* a physical system when the numbers share the same *structure* as the physical system. We'll work up to this idea.

2.1 Using numbers to represent quantities

Suppose we want to assign numbers to physical objects, in such a way that the numbers represent how much mass those objects have. We must choose some particular scale when we make these numerical assignments. For instance, if we choose the kilograms scale, a certain object, O, might get assigned the number 5; whereas if we choose the grams scale, O will get assigned the number 5000.

However, to understand more deeply what is involved in choosing a numerical scale, we need to take a few steps back, and move more slowly.

Consider this assignment of numbers to massive objects:

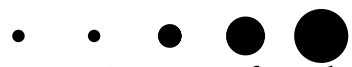


(The sizes in the diagram represent how massive the objects are.) This might seem like a silly assignment. After all the biggest number it assigns, 167, isn't the most massive object. However, it's not *entirely* silly. In the diagram, two objects have the same mass, the second and the fifth. And in fact, they are the only objects that are assigned the same number. So we can say the following about this assignment:

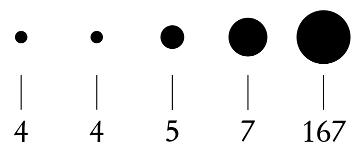
(1) x is assigned the same number as y iff x and y have the same mass

Thus, the assignment of numbers *reflects certain facts about mass*. Put another way, certain facets of the assignment are *physically significant*: namely: identity and distinctness of numbers assigned. Given this, there is certain information about the masses of objects we can recover, if someone tells us the numbers assigned to objects: namely, information about the *same-mass-as* relation.

So, the assignment isn't totally silly; it reflects some of the facts about mass. However, there are more facts about mass beyond merely facts to the effect that a given pair of objects do, or do not, have exactly the same mass. For example, some things are more massive than others; you can line up the objects from least to most massive:



Accordingly, we might choose an assignment of numbers that reflects this ordering of the massive objects:



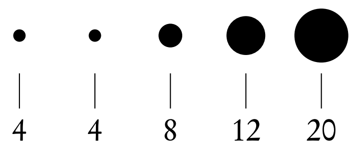
This assignment of numbers "reflects" the facts about which objects are more massive in the sense that it obeys this principle:

(2) x is assigned a greater number than y iff x is more massive than y

This assignment encodes more facts about mass than the preceding one. It now encodes the more-massive-than relation, in addition to encoding the has-exactly-as much-mass relation.

Here's another way to think about it. The numbers we are using to represent mass have various numerical features. There is of course an identity relation over the numbers; but there's also a relation being-a-greater-number-than. In moving to this second assignment, we're taking advantage of more of the numerical features: in the first assignment, the greater-number-than relation over numbers wasn't used to represent anything about mass, but it is used in the second assignment.

We're still not using all the features of the numbers that we can. The smallest two objects aren't *that* much smaller than the largest object, but they're getting assigned a *much* smaller number. We're only using the ordering of the numbers at this point, not their sizes. But for mass, more than just the ordering is significant; there are also certain facts about how much bigger one mass is than another. For example, in the diagram, the third object appears to be about as massive as the first two combined. Let's stipulate that this is indeed the case, and further, that this pattern continues (as it appears to): namely, let's assume that each mass in the diagram starting with the third is exactly as massive as the two preceding masses combined. We could reflect this in an assignment like the following:



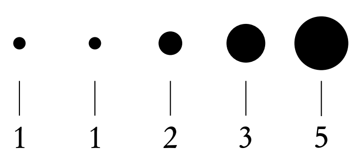
This assignment has the following property, in addition to the properties (1) and (2):

(3) The sum of the numbers assigned to x and y equals the number assigned to z iff x and y's combined masses equal z's mass

Now we're using facts about the sums of assigned numbers to code up facts about this three-place mass relation: *x and y's combined masses equals z's mass*.

Note: this name I'm giving to this relation may suggest that to say that x and y's combined masses equals z's mass is to say that in some underlying numerical scale, if you add the numbers assigned to x and to y together, you get the number assigned to z. But that's not the idea. The idea is rather that this is simply a three-place relation between objects that is not defined in terms of numbers at all. (It doesn't hold "in virtue of" facts about numerical assignments.) Indeed, it is a relation you could measure directly: put the objects x and y on one side of a pan balance, put z on the other, and see if they balance.[2]

Notice that the last assignment isn't the only one that satisfies (1), (2), and (3). Here is another one:



So to sum up: we can assign numbers to objects in a way that encodes information about the objects' nonnumeric properties. There are different sorts of information that can be encoded (recall (1), (2), and (3)). And multiple assignments can encode a given sort of information.

[2] Well, this works if x and y don't overlap. But the idea is supposed to be that if $C(x, x, y)$ then y's mass is double x's. So the real way to test whether $C(x, y, z)$ is to put three nonoverlapping objects with the *same masses* as x, y, and z, respectively, on the pan balance—where two objects have the same mass iff each is at least as massive as the other.

2.2 Relational structures, homomorphisms and representation theorems

Let's generalize some of these ideas. Suppose we're trying to represent some nonnumeric facts using numbers.

- Think of the nonnumeric facts—such as the nonnumeric facts about mass—as a *relational structure*: an n-tuple $\langle A, R_1 \dots R_n \rangle$, where A is a set and $R_1 \dots R_n$ are relations on that set. In the example of mass, the initial relational structure we chose had no relations, and just the set A of the five massive objects. The second relational structure included the two-place relation $\succeq$ of being at-least-as-massive-as; and the third relational structure included both $\succeq$ and the three-place relation C of combining-to-equal-in-mass: $\langle A, \succeq, C \rangle$. (Actually earlier I was considering a closely related relation, that of being strictly more massive than.)

- Think of the mathematical entities we're using as another relational structure. In the case of the third assignment, the mathematical structure would be $\langle \mathbb{R}, \geq, + \rangle$, where $\mathbb{R}$ is the set of real numbers, $\geq$ is the greater-than-or-equal-to relation on those numbers, and $+$ is the addition relation on real numbers: $+(x, y, z)$ holds iff $x + y = z$.

- A mathematical structure $\langle B, S_1 \dots S_n \rangle$ will be useful tool to represent a nonmathematical structure $\langle A, R_1 \dots R_n \rangle$ if it contains a "homomorphic image" of that nonmathematical structure—iff there is some function f (a "homomorphism") from A into B such that for each R_i, $R_i(x_1 \dots x_m)$ iff $S_i(f(x_1) \dots f(x_m))$. For example, the function indicated by the lines in the third example is a homomorphism from $\langle A, \succeq, C \rangle$ into $\langle \mathbb{R}, \geq, + \rangle$. (Needn't be an ismomorphism—two objects can have the same mass and thus get assigned to the same number.)

- The basic idea is that homomorphic structures have analogous structure. If we have a homomorphism, we can use it to pass from information about the mathematical structure to information about the nonmathematical structure. For example, let f be the homomorphism from $\langle A, \succeq, C \rangle$ into $\langle \mathbb{R}, \geq, + \rangle$ that was discussed above (it assigns the values $4, 4, 8, 12, 20$ to the masses depicted; call them a, b, c, d, e). Simple arithmetic tells us that $+(8, 12, 20)$; but then since $f(c) = 8, f(d) = 12, f(e) = 20$, we can infer from the fact that f is a homomorphism that $C(a, b, c)$. A homomorphism, in fact, is just a particular scale.

The main thing measurement theory does is prove stuff about these homomorphisms. For example, it shows how prove that if a nonmathematical structure has certain features, then there exists at least one homomorphism from it into an appropriate mathematical structure. Theorems to the effect that such homomorphisms exist are called "representation theorems".

2.3 Uniqueness theorems

We saw that the homomorphisms we were discussing aren't unique. In the example of the mass series, the function assigning $4,4,8,12,20$ was a homomorphism; but so was the function that assigned $1,1,2,3,5$. This corresponds to the fact that a choice of scale is arbitrary.

One thing we want to know is "how unique" the homomorphisms—scales–are. What we would expect, in the case of mass, is that every scale would be a constant multiple (often called a similarity transformation) of every other: i.e., if f and g are homomorphisms from $\langle A, \succeq, C\rangle$ to $\langle \mathbb{R}, \geq, +\rangle$, then for some positive real number k, for all $x \in A$, $f(x) = kg(x)$. Proving this fact is called proving a "uniqueness theorem".

Suppose all the homomorphisms from the nonmathematical to the mathematical structure are similarity transformations of each other. Then we say we have a "ratio scale", because even though the scales (homomorphisms) assign different absolute values, they all assign the same ratios. For let f and g be any scales and x and y be any two massive objects; then:

$$\frac{f(x)}{f(y)} = \frac{kg(x)}{kg(y)} = \frac{g(x)}{g(y)}$$

One kind of uniqueness theorem, then, would say that any two homomorphisms from a certain nonmathematical structure into a certain mathematical structure are similarity transformations of each other. There are other kinds of uniqueness theorems one could prove. (Which kind of uniqueness theorem can be proved depends on the features of the relational structures in question.) For some pairs of mathematical and nonmathematical structures, the uniqueness theorem says that any two homomorphisms f and g are affine transformations, in that for some constants $k > 0$ and a, $f(x) = kg(x) + a$ (for all x). In these cases we call the scale an "interval scale", since all such functions agree on ratios of intervals, in that for any x and y,

$$\frac{f(x_1)-f(x_2)}{f(y_1)-f(y_2)}=\frac{g(x_1)-g(x_2)}{g(y_1)-g(y_2)}$$

The big difference here is that which element is assigned to the number zero by an interval scale is physically insignificant. So this would be appropriate for temperature (ignoring absolute zero). Which element is assigned to zero *is* physically signiciant in the case of mass, since there are no negative masses.

Another case (here the homorphisms are much *less* unique): an *ordinal scale* is one where all that is preserved by the homomorphisms is order—

$$f(x)>f(y) \text{ iff } g(x)>g(y)$$

To summarize:

Scale type	Preserves	Transformations
Ratio	ratios	similarity ($f=kg$)
Interval	ratios between intervals	affine ($f=kg+a$)
Ordinal	order	monotone

2.4 Assumptions made

Representation and uniqueness theorems need to make certain assumptions about the nonmathematical structure in question. For example, take the example of $\langle A,\succeq,C\rangle$ and $\langle \mathbb{R},\geq,+\rangle$. In order to prove that these are homomorphic, you're going to need to assume that the relation $\succeq$ is transitive. Why? Because $\geq$ is transitive. Suppose that there does exist a homomorphism f; and suppose that $x\succeq y$ and $y\succeq z$. By the definition of homomorphism, it follows that $f(x)\geq f(y)$ and $f(y)\geq f(z)$. But by the transitivity of $\geq$, $f(x)\geq f(z)$; and then by the definition of homomorphism, $x\succeq z$. What we just showed is that if there exists any homomorphisms at all, then $\succeq$ is transitive. So if $\succeq$ *isn't* transitive, then there can't exist any homomorphisms.

Here is an example in which the failure of such an assumption means that you can't have a representation theorem (at least, of the sort we've been discussing). Consider representing the painfulness-to-me of certain pains, and in particular the structure $\langle P,R\rangle$ where P is the set of my pains and R is the relation of being more painful than. Is there a homomorphism from this structure to $\langle \mathbb{R},>\rangle$? You might think: sure, since it's plausible that R is transitive. But in order for there to be a homomorphism, R needs to be more than transitive: it also needs to be "negatively transitive", i.e., it needs to be that if $\sim Rxy$ and $\sim Ryz$ then $\sim Rxz$. The reason is that $>$ is negatively transitive (the negation of $>$ is just $\leq$, which is transitive). And it's arguable that R isn't negatively transitive. Consider a series of painful punches to my stomach, $p_1 \dots p_n$, in which adjacent members vary in force to such a small degree that I can't tell them apart, but in which the last member is definitely less painful than the first. Let p_i and p_{i+1} be adjacent members. Since I can't tell them apart, it would seem that $\sim Rp_i p_{i+1}$. So if $\sim R$ is transitive, it follows that $\sim Rp_1 p_n$. But that's false; $Rp_1 p_n$. So: there just can't be a homomorphism. If R isn't negatively transitive, then you just can't use numbers to represent the more-painful-than relation over pains, in the sense that a higher number is to be assigned if and only if the pain is greater.

Here's another example. Suppose that there existed massive objects of each finite mass, but that there also existed an infinitely massive object—infinite in the sense that it is more massive than each of the finitely massive objects. In that case there again couldn't be a representation theorem (of the type we've been talking about). For the homomorphism would need to "use up" all the real numbers in its assignments to all the finite things; but then there would be no number left to assign to the infinite thing.[3] The assumption that there are no masses "at infinity" is usually called an "Archimedean" assumption: that any mass can be reached or exceeded from any other "by a finite number of steps". One such assumption may be set up as follows. First we need to define the idea of x being "n times as massive as" y:

$$
\begin{aligned}
M^n xy =_{\text{df}} & \text{ for some } y_1 \dots y_n : \\
& y_1 = y, \\
& C(y, y_i, y_{i+1}) \text{ for } 1 \leq i < n, \text{ and} \\
& y_n = x
\end{aligned}
$$

Here, then, is the Archimedean assumption:

> For any x and y, if $x \succeq y$ then for some positive integer n and some z, $M^n zy$ and $z \succeq x$

So: representation and uniqueness theorems are proved, for particular nonmathematical and mathematical structures obeying certain assumptions.

[3] To make this argument rigorous, you'd need to refine the statement of the example: it would need to say that the Archimedean assumption is obeyed for the finite things.

Here are some typical assumptions for a quantity like mass (these may not be sufficient to prove the representation and uniqueness theorems; I haven't checked):

- $\succeq$ is transitive and strongly connected (i.e. $x \succeq y$ or $y \succeq x$ holds for each x and y)
- C is "commutative" and "associative" in that:
  if $C(x,y,a)$ then $C(y,x,a)$
  if $C(x,y,a)$ and $C(a,z,b)$ and $C(y,z,c)$ then $C(x,c,b)$
- Adding the same mass preserves $\succeq$, in that:
  if $x \succeq y$, and if $C(x,z,x')$ and $C(y,z,y')$, then $x' \succeq y'$
- if $C(x,y,z)$ then $z \succ x$ (mass is never negative)
- Archimedean assumption
- Existence of multiples: for each x and integer n, there exists some y such that $M^n yx$

2.5 Sketch of proofs

Proving representation theorems and uniqueness theorems can be tricky, but it's nice to have a rough idea of how it goes. So I'll sketch parts of the proofs in the case of the nonmathematical structure $\langle A, \succeq, C\rangle$ for mass discussed above, and the mathematical structure $\langle \mathbb{R}, \geq, +\rangle$.

In the proofs, it will be helpful to use the defined notion of one object being *more* massive than another. This can be defined thus:

$$x \succ y =_{\text{df}} y \not\succeq x$$

(I'll also sometimes write $x \preceq y$ or $x \prec y$ in place of $y \succeq x$ or $y \succ x$, respectively.)

We'll also need to use various facts. Of course we'll need to use the definition of a homomorphism, and the various assumptions about the mass structure. We'll also need to use some facts that can be proved from these. For example, just as homomorphisms must "respect" $\succeq$ and C, they must also also respect the defined relations M^n and $\succ$:

$$\begin{aligned} &M^n xy \text{ iff } g(x) = n g(y) \\ &x \succ y \text{ iff } g(x) > g(y) \qquad \text{(for any homomorphism } g\text{)} \end{aligned}$$

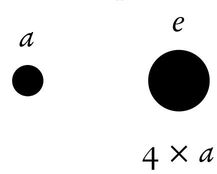
(To see that the first is true, note that given the definition of "$M^n xy$", and given how homomorphisms must interact with C facts, $g(x) = \underbrace{g(y) + g(y) \cdots + g(y)}_{n \text{ times}}$.)

OK, here is a sketch of how to prove the representation theorem. We must prove that there exists at least one homomorphism f from the mass structure into the real numbers structure. The proof has two halves. In the first half, we construct a certain function f, and in the second half, we show that f is a homomorphism. I'm only going to do the first half.

The first step in constructing f is to arbitrarily pick some object $e \in A$ that will function as the unit. So we'll let $f(e) = 1$.

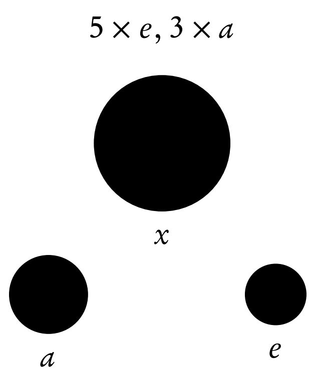
Now take any other $a \in A$. What should we set $f(a)$ to be? As we'll see, we no longer have any freedom in this. If we want f to be a homomorphism, the fact that we let $f(e) = 1$ determines what we must have f assign to all other objects.

If a happens to be exactly n times as massive as e, for some integer n (i.e. $M^n ae$) then since homomorphisms must respect M^n, we must let $f(a) = nf(e) = n$. Similarly, if e just happens to be n times as massive as a, then we must let $f(a) = \frac{1}{n}$. For example:



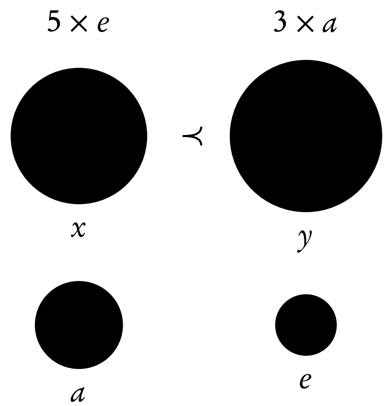
so $f(a) = \frac{1}{4}$

Even if neither of these cases holds, so that neither a nor e is a "multiple" of the other, there may happen to exist some third mass that is a multiple of both. That is, perhaps for some $x \in A$, and some integers m and n, $M^m xe$ and $M^n xa$. Then we must set $f(a) = \frac{m}{n}$. (Because $nf(a) = f(x) = mf(e) = m$.) For example:



so $f(a) = \frac{5}{3}$

It might be that none of the above holds. (This happens when a's mass is measured by an irrational number, relative to our choice of e as the unit.) But even then, the choice of $f(a)$ is determined. To work up to this, suppose that for some integers m and n, something that is m times e is *smaller* than something that is n times a. That is, for some x and y, $M^m xe$, $M^n ya$, and $x \prec y$. Then $\frac{m}{n}$ is below what we must set $f(a)$ to. (This is intuitively true, but here's why it's true: since $M^m xe$ and $M^n ya$, $f(x) = m$ and $f(y) = nf(a)$; so $\frac{m}{n} = f(a)\frac{f(x)}{f(y)}$; but since $x \prec y$, $f(x) < f(y)$ and so $\frac{f(x)}{f(y)} < 1$.) For example:



so $f(a) > \frac{5}{3}$

So what we've seen so far is this: when m times e is smaller than n times a, $\frac{m}{n}$ will be less than what we must make $f(a)$. But notice that by appropriate choices of m and n, we can make $\frac{m}{n}$ get closer and closer to what $f(a)$ must be: we make m times e be closer and closer (albeit still smaller) than n times a. Though I won't prove this, it turns out that the limit (least upper bound) of all such fractions $\frac{m}{n}$ is what we must set $f(a)$ to.

(By the way, it's intuitively easy to see where the Archimedean assumption will be needed in a proper proof. If a were "infinitesimally small" relative to e, then there couldn't be m and n such that m times e is smaller than n times a.)

So that is the sketch of the construction of f. The next step, were we continuing, would be to show that f is a homomorphism.

What about the uniqueness theorems? We want to show that any two homomorphisms are scalar multiples of each other. The way we do it is to show that any homomorphism g is a scalar multiple of the homomorphism f that we constructed earlier—i.e., that for all $a \in A$, $g(a) = kf(a)$ for some constant real number k. How should we choose the constant k? Well, k needs to equal $\frac{g(a)}{f(a)}$ for all a if we're to succeed; but $f(e) = 1$; so k must be $g(e)$.

So, let's show that $g(a) = g(e)f(a)$, i.e., that $\frac{g(a)}{g(e)} = f(a)$ for all a. Suppose for reductio that $\frac{g(a)}{g(e)} \neq f(a)$. Then either $\frac{g(a)}{g(e)} < f(a)$ or $\frac{g(a)}{g(e)} > f(a)$. Let's consider the first case (the proof in the case of the second is parallel.) Between any two real numbers there is some rational number; so there are integers m and n such that $\frac{g(a)}{g(e)} < \frac{m}{n} < f(a)$. Now, let's choose an object x whose mass is m times that of e, and an object y whose mass is n times that of a. That is, $M^m xe$ and $M^n ya$. (Note the use of the existence of multiples.) Thus $g(x) = mg(e)$, and $g(y) = ng(a)$. So $\frac{g(a)}{g(e)} = \frac{mg(y)}{ng(x)}$; and so, since $\frac{g(a)}{g(e)} < \frac{m}{n}$, we know that $\frac{g(y)}{g(x)} < 1$ and so $y \prec x$. But since $M^m xe$ and $M^n ya$, $f(x) = mf(e)$ and $f(y) = nf(a)$, and so:

$$\frac{\frac{m}{n}}{f(a)} = \frac{f(x)}{f(y)}$$

But the left hand side of this is less than 1 (since $\frac{m}{n} < f(a)$) whereas the right hand side is greater than 1 (since $y \prec x$); contradiction.

2.6 Kinds of quantities

We've seen how to prove representation and uniqueness theorems for mass. Quantities other than mass might obey similar theorems. For example, suppose that instead of dealing with massive objects, we were instead dealing with measuring rods. We would have a binary relation of at-least-as-long-as, and a three-place relation of concatenation: one rod is as long as two others laid end-to-end. These relations might obey exactly parallel assumptions to those obeyed by $\succeq$ and C. And so one could prove the same uniqueness and representation theorems (since those theorems depend only on the structure of the assumptions).

The theorems relied essentially on there being the mass relations C and $\succeq$: on it making sense to speak of one object being at least as massive as another, and of one object being the combined mass of two others. These seem like sensible assumptions to make about mass, but for other quantities, parallel assumptions aren't justified. Take wit, for example. Perhaps it makes sense to speak of one person being at least as witty as another. But it surely makes no sense to speak of one person as being exactly as witty as two other people combined. What this means is that we can't have a representation theorem for wit of the same sort as the one we had for mass. The one for mass relied essentially on using the relation C to constrain the homomorphisms. But we may be able to prove a different kind of representation theorem. If the at-least-as-witty relation has certain appropriate features, then we may be able to prove that the structure $\langle P, \succeq_W \rangle$ ($P =$ the set of people; $\succeq_W =$ the at-least-as-witty relation) is homomorphic to $\langle \mathbb{R}, \geq \rangle$, and that all homomorphisms have the same order. The size of the numbers assigned would not be significant; all that would be significant is their order.

It could be even worse. Maybe the $\succeq_W$ relation isn't connected—that is, maybe there are two people that are "incomparable" in terms of wit in the sense that neither is at least as witty as the other. In that case we won't be able to have such homomorphisms (because $\geq$ *is* connected: for any real numbers a and b, either $a \geq b$ or $b \geq a$). But we could still have a kind of representation theorem: maybe $\langle P, \succeq_W \rangle$ is homomorphic to some mathematical structure other than $\langle \mathbb{R}, \geq \rangle$ (some structure that isn't linearly ordered). That wouldn't be much of a representation, but still.

2.7 Measurement theory: metaphysics and epistemology

Measurement theory was largely developed by philosophers of science who were concerned with questions like: we can't *observe* correlations between physical objects and real numbers, so how can the use of real numbers be justified in terms of things we *can* observe?

But metaphysicians also have concerns about quantity (as we noted earlier), and they too can answer them using measurement theory. Recall how we introduced two main concerns about quantities. First, how can numbers be so useful in science, when the fundamental facts about the quantities don't involve numbers at all? And second, what does it mean to say that certain kinds of claims, such as the claim that some object has mass 2, or that a certain person is exactly twice as witty as another, don't "make sense"? There are natural ways to answer these questions using measurement theory.

To the first, we could say that the relations in the nonmathematical structures ($\succeq$ and C in the case of mass) are fundamental relations. We could then use the representation and uniqueness theorems to show how numbers could be useful in science, even though the fundamental mass relations have nothing to do with numbers. What we do when we use numbers to talk about a quantity is: we pick one of the homomorphisms, and use it to talk about objects. Talk of the numbers assigned by such a homomorphism carries with it information about the purely nonmathematical structure, as we saw earlier.

As for the second: as we saw, there is more than one homomorphism from

$\langle A, \succeq, C \rangle$ into $\langle \mathbb{R}, \geq, + \rangle$. So if you just say "the mass of an object is 2", which homomorphism are you talking about? Now, even when you say "the ratio of x's mass to y's mass is 1.75", there still is the problem that you haven't mentioned a specific homomorphism. But here it doesn't matter as much, since the statement doesn't vary in truth value from homomorphism to homomorphism. (One could have said instead: "on every homomorphism, the ratio of x's mass to y's mass is 1.75".) What about wit? Why doesn't it make sense to say "twice as witty", and more generally to measure wit with real numbers? This is a little tricker, but the crucial thing is that there is no metaphysical basis for a ratio scale of wit, in that: there are *no* wit relations such that a wit structure is homomorphic to $\langle \mathbb{R}, \geq, + \rangle$. Now, this is tricky, for there may well be relations ("abundant" relations in the sense of Lewis (1986, pp. 59–69)) over people with the right formal properties. But none of these is "distinguished". (The issue is tricky because since no wit relations are *perfectly* fundamental, 'distinguished' can't mean perfectly fundamental.)

Metaphysicians and philosophers of science have different concerns, which can lead them in different directions here. Just a few examples. First, notice that relations like $\succeq$ and C are comparative. They don't specify particular masses of their relata, they only specify relative mass relations. It's very natural to focus on such relations if you're concerned with epistemology, because relative mass relations are the ones we directly measure. But it's less clear whether it's plausible to take such comparisons to be fundamental relations. After all, doesn't the fact that x is at least as massive as y hold in virtue of the particular masses of x and y?

To take another example, the representation and uniqueness theorems make certain existence assumptions. The uniqueness theorem, for example, makes essential use of the assumption of the existence of multiples. There is a kind of epistemic concern here: how do we know that we can always find such multiples? But there is a more pressing concern: is the metaphysical assumption that there exist all these multiples justified? Couldn't I have exactly the same mass as I actually have, even if there didn't exist arbitrarily large multiples?

To take a final example, philosophers of science often worried that certain of the assumptions, such as the transitivity of $\succeq$, are unjustified given experimental error. Suppose the idea of $x \succeq y$ is "if you put x and y on a pan balance, the y side won't move downward". And suppose your pan balance won't register differences smaller than a certain amount. Then this relation will be intransitive. But this sort of concern doesn't seem to have any metaphysical analog.

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